PH108 (Division 3) Lectures on TUESDAY & FRIDAY 1400-1525
SLOT NO 10A + 10B LA 101

Instructor (D3): Kantimay Das Gupta : kdasgupta@phy

Reference texts:

D J Griffiths : Introduction to electromagnetism
Feynman Lectures: vol 2

Vector Analysis (Schaum series) M Spiegel
Mathematical methods : Pipes & Harvil

Several other classic texts:
Panofsky and Philips
J D Jackson

ATTENDANCE : 80% REQUIRED
EVALUATION Quiz1=15 : Midsem=35 : Quiz2=10 : Endsem=40

Electromagnetism

Basic principles known for about 150 years.

Mature subject with a well defined structure.

Regime of validity well undesrtood.

Great success: explaining propagation & generation of electromagnetic radiation,
Forces of adhesion and cohesion.

First example of a classical field theory....particles and fields both carry energy and
Momentum

Fails when we go to atomic scale

Gravity and electromagnetism are markedly different too, though both
have " inverse square force“ laws.

Two key questions:

Why do we use vectors?
Why do we use many co-ordinate systems?

Symmetry of the problem and the shape of the objects involved must be taken into account.

What is a field ? What are the typical questions one asks?

A quantity defined or measured over a certain area/volume of space.

Scalar field Temperature defined over a region $T(x,y,z)$

Vector field Electric, Magnetic field : $\mathbf{E}(x,y,z)$ $\mathbf{B}(x,y,z)$
velocity of water $\mathbf{v}(x,y,z)$ in a pipe, river, ocean

Matrix/Tensor field Stress, Strain inside a material like a concrete beam.
With every point a matrix like object is associated.

A field is also like an object with a large number of degrees of freedom.

How is the field created? What is the "source" ?
How does the field affect particles in it (Interaction of field with matter)?

A systematic way of handling co-ordinate systems : Part 1

Many types of co-ordinates are needed, so that we can use the natural symmetry of a problem.

Equations would have the simplest form and minimum number of free variables if the co-ordinate system is chosen intelligently.

How to define a co-ordinate system?

Few typical systems:

Plane Polar
Spherical Polar
Cylindrical

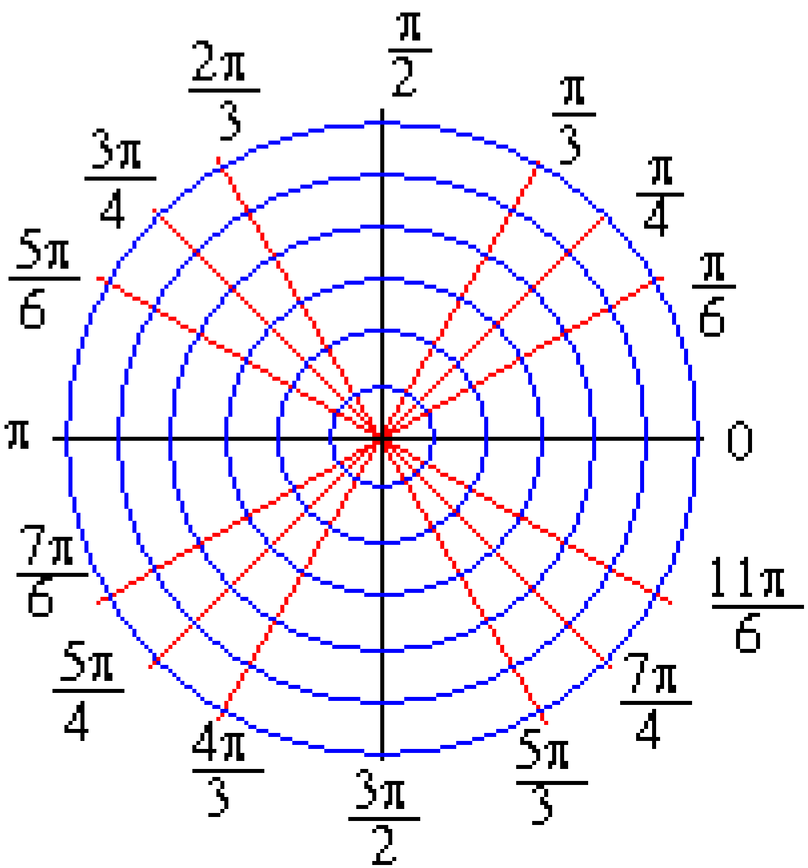
Then how to define your own if you need?

Plane Polar (r,θ) in detail

STEP 1: Write down the relation with (x,y) co-ordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

STEP 2: Draw the co-ordinate grid



How do r=constant lines look?

How do θ= constant lines look?

Plane Polar (r,θ) in detail

STEP 3: What happens when the independent variables are changed infinitesimally

$$\begin{aligned} \delta x &= \cos\theta\,\delta r - r\sin\theta\,\delta\theta \\ \delta y &= \sin\theta\,\delta r + r\cos\theta\,\delta\theta \end{aligned}$$

STEP 4: Which direction would we move, if only one variable was changed?

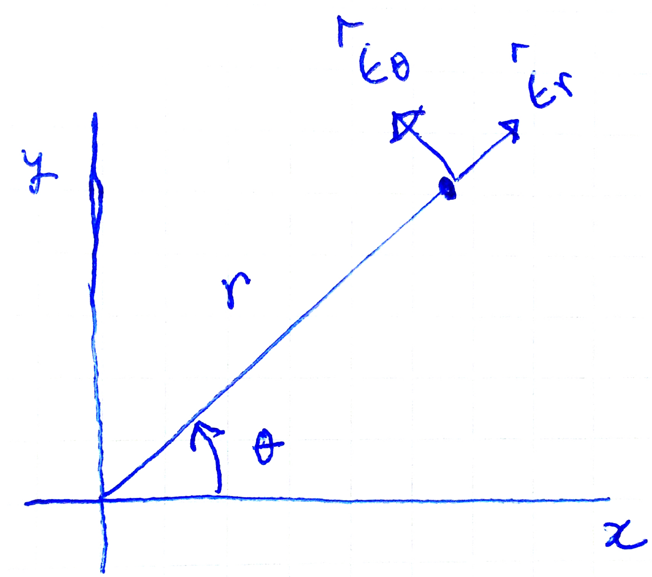
$\underline{\delta\theta = 0}$

$$\begin{aligned} \boldsymbol{i}\,\delta x + \boldsymbol{j}\,\delta y &= (\boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta)\,\delta r \\ &= \boldsymbol{\epsilon}_r\,\delta r \end{aligned}$$

$\underline{\delta r = 0}$

$$\begin{aligned} \boldsymbol{i}\,\delta x + \boldsymbol{j}\,\delta y &= (-\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta)\,r\,\delta\theta \\ &= \boldsymbol{\epsilon}_\theta\,r\,\delta\theta \end{aligned}$$

$$\delta\boldsymbol{r} = \boldsymbol{\epsilon}_r\,\delta r + \boldsymbol{\epsilon}_\theta\,r\,\delta\theta$$



$$\boldsymbol{\epsilon}_r \cdot \boldsymbol{\epsilon}_\theta = 0$$

Curvilinear but still orthogonal

Plane Polar (r,θ) in detail

STEP 5: What happens to the element of area?

i.e take a small step in the $\boldsymbol{\epsilon}_r$ direction and a small step in the $\boldsymbol{\epsilon}_\theta$ direction
What is the "infinitesimal" area enclosed by these two perpendicular vectors?

$$
\begin{aligned}
dA &= \left| \boldsymbol{\epsilon}_r \times \boldsymbol{\epsilon}_\theta \right| \\
&= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} \delta\theta\,\delta r \\
&= r\,\delta\theta\,\delta r
\end{aligned}
$$

STEP 6: What happens to the element of distance or arclength?

$$
\begin{aligned}
ds^2 &= \delta \boldsymbol{r} . \delta \boldsymbol{r} \\
&= dr^2 + r^2 d\theta^2
\end{aligned}
$$

In orthogonal co-ordinates there will be no cross terms
in the arclength expression.

Plane Polar (r,θ) in detail

STEP 7: Now suppose a SCALAR function of co-ordinates is defined (like Temperature over a region), T(x,y,z)

We change our position by a small VECTOR $\delta \boldsymbol{r}$, and ask dT = ?

We want a function such that :

$$\begin{aligned} \delta T &= \frac{\partial T}{\partial r}\delta r + \frac{\partial T}{\partial \theta}\delta\theta \\ &= [some \quad fn].\delta \boldsymbol{r} \\ &= [some \quad fn].(\boldsymbol{\epsilon}_r \delta r + \boldsymbol{\epsilon}_\theta r\delta\theta) \end{aligned}$$

$$[some \quad fn] = \boldsymbol{\epsilon}_r \frac{\partial T}{\partial r} + \boldsymbol{\epsilon}_\theta \frac{1}{r}\frac{\partial T}{\partial \theta}$$

$$The \quad combination \quad is \quad called \quad gradient$$

$$\boldsymbol{\nabla} = \boldsymbol{\epsilon}_r \frac{\partial}{\partial r} + \boldsymbol{\epsilon}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}$$

Gradient is the generalisation of the derivative in 1 dimension

Prove :

grad T is perpendicular to surfaces of constant T

What form would grad T take in cartesian co-ordinates?

Plane Polar (r,θ) in detail

STEP 8: What are velocity and acceleration components, when a particle's motion
Is described using polar co-ordinates?

$$\boldsymbol{v} = \frac{\delta \boldsymbol{r}}{\delta t}$$
$$= \frac{\delta}{\delta t}\left(\boldsymbol{\epsilon}_r \,\delta r + \boldsymbol{\epsilon}_\theta \, r\,\delta\theta\right)$$
$$= \boldsymbol{\epsilon}_r \frac{dr}{dt} + \boldsymbol{\epsilon}_\theta \, r \frac{d\theta}{dt}$$
$$\boldsymbol{a} = \frac{d}{dt}\left(\boldsymbol{\epsilon}_r \frac{dr}{dt} + \boldsymbol{\epsilon}_\theta \, r \frac{d\theta}{dt}\right)$$

Unlike cartesian unit vectors the unit vectors here are not constant and must be differentiated themselves.

$$\dot{\boldsymbol{\epsilon}}_r = ?$$
$$\dot{\boldsymbol{\epsilon}}_\theta = ?$$

Using our result from STEP 4...

$$\begin{pmatrix} \boldsymbol{\epsilon}_r \\ \boldsymbol{\epsilon}_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \end{pmatrix}$$

hence

$$\begin{pmatrix} \dot{\boldsymbol{\epsilon}}_r \\ \dot{\boldsymbol{\epsilon}}_\theta \end{pmatrix} = \dot{\theta} \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \end{pmatrix}$$

Plane Polar (r,θ) in detail

STEP 8: What are velocity and acceleration components, when a particle's motion Is described using polar co-ordinates?

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$$= \frac{\delta}{\delta t}\left(\boldsymbol{\epsilon}_r \delta r + \boldsymbol{\epsilon}_\theta r \delta\theta\right)$$

$$= \boldsymbol{\epsilon}_r \frac{dr}{dt} + \boldsymbol{\epsilon}_\theta r \frac{d\theta}{dt}$$

$$\boldsymbol{a} = \frac{d}{dt}\left(\boldsymbol{\epsilon}_r \frac{dr}{dt} + \boldsymbol{\epsilon}_\theta r \frac{d\theta}{dt}\right)$$

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hence

$$\begin{pmatrix} \dot{\boldsymbol{\epsilon}}_r \\ \dot{\boldsymbol{\epsilon}}_\theta \end{pmatrix} = \dot{\theta} \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \end{pmatrix}$$

Plane Polar (r,θ) in detail

Using the last two results:

$$\begin{pmatrix} \dot{\boldsymbol{\epsilon}}_r \\ \dot{\boldsymbol{\epsilon}}_\theta \end{pmatrix} = \dot{\theta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\epsilon}_r \\ \boldsymbol{\epsilon}_\theta \end{pmatrix} \qquad \begin{cases} \dot{\boldsymbol{\epsilon}}_r = \dot{\theta}\,\boldsymbol{\epsilon}_\theta \\ \dot{\boldsymbol{\epsilon}}_\theta = -\dot{\theta}\,\boldsymbol{\epsilon}_r \end{cases}$$

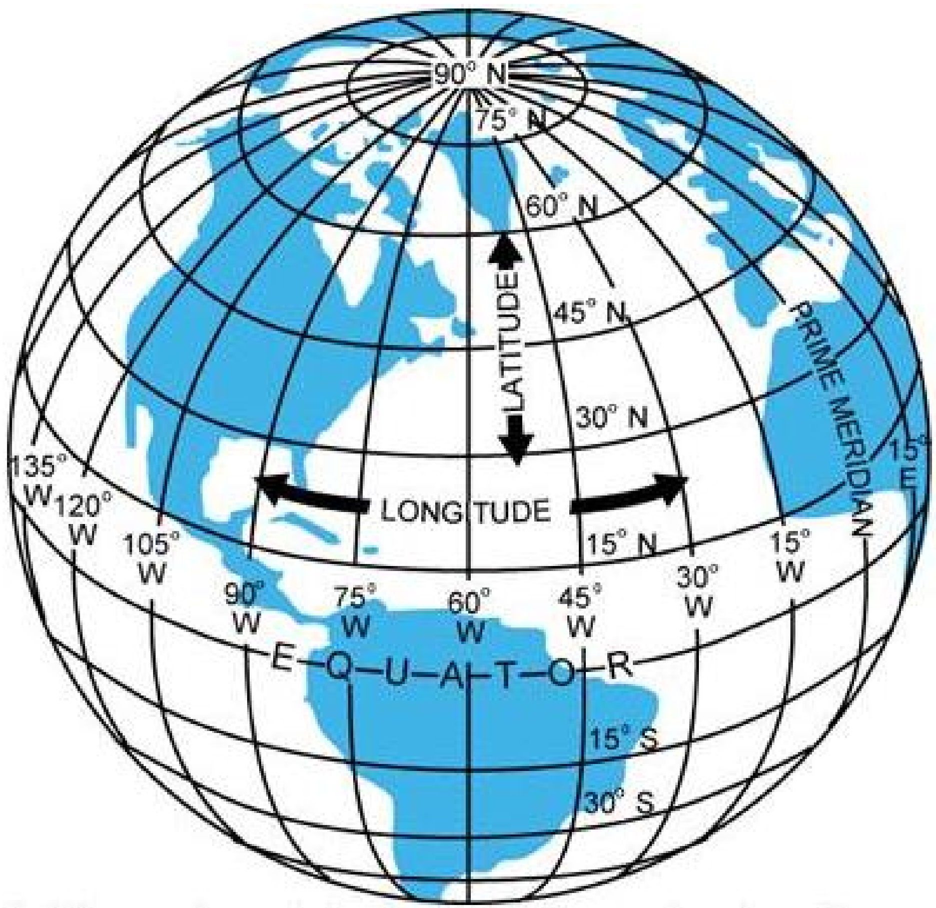
$$\begin{aligned} \boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} &= \frac{d}{dt}\left(\boldsymbol{\epsilon}_r \dot{r} + \boldsymbol{\epsilon}_\theta r \dot{\theta}\right) \\ &= \boldsymbol{\epsilon}_\theta \dot{\theta}\, \dot{r} + \boldsymbol{\epsilon}_r \ddot{r} - \boldsymbol{\epsilon}_r \dot{\theta} r \dot{\theta} + \boldsymbol{\epsilon}_\theta \dot{r} \dot{\theta} + \boldsymbol{\epsilon}_\theta r \ddot{\theta} \\ &= \left(\ddot{r} - \dot{\theta}^2 r\right)\boldsymbol{\epsilon}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\boldsymbol{\epsilon}_\theta \end{aligned}$$

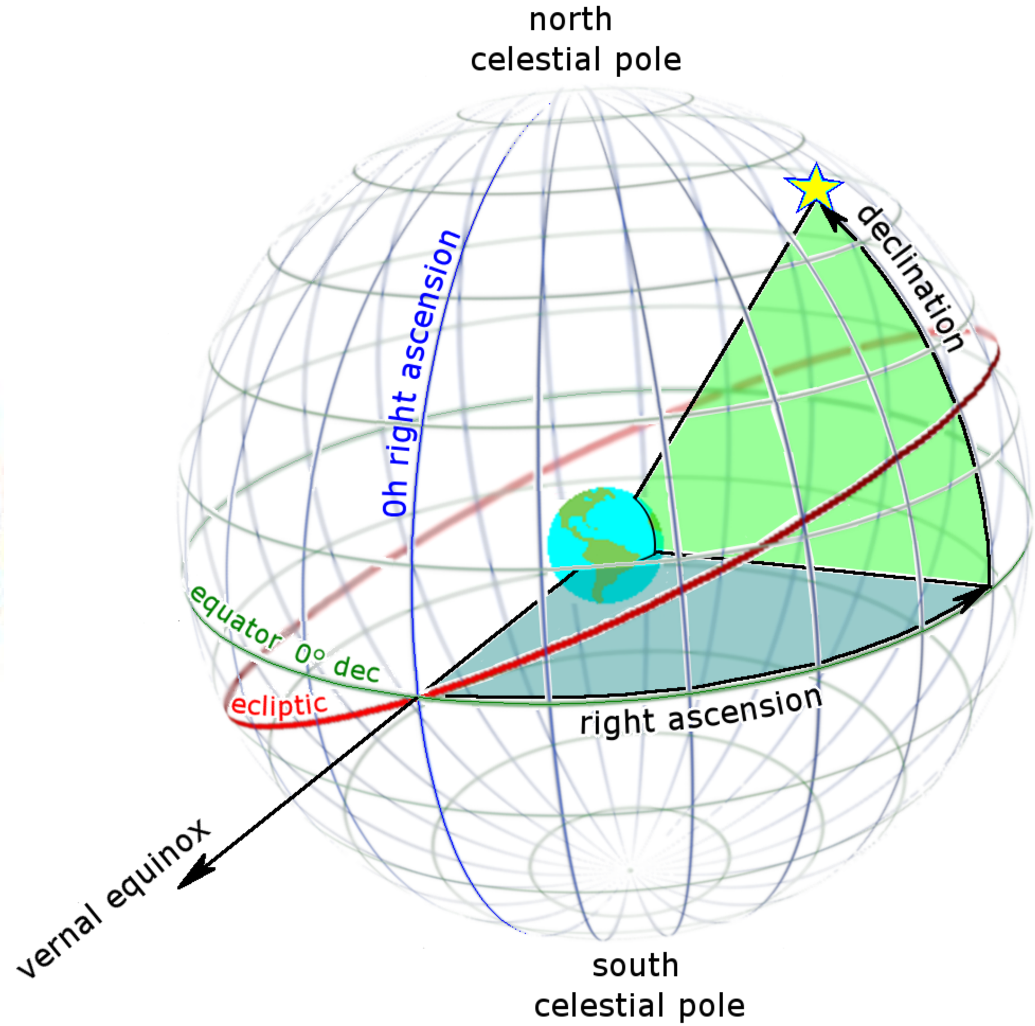
What are the physical meanings of the various terms in the result for acceleration?

If the force on the particle is "central", then which quantity is conserved?

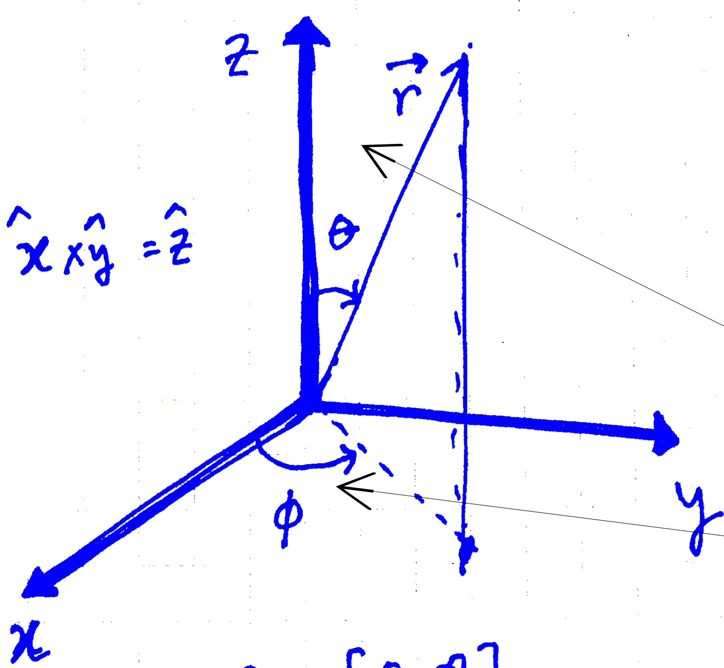
You are given an arbitrary vector in cartesian $\left(\boldsymbol{i}F_x + \boldsymbol{j}F_y\right)$.
How will you go over to $\left(\boldsymbol{\epsilon}_r F_r + \boldsymbol{\epsilon}_\theta F_\theta\right)$?
What can you say about the matrix connecting the two sets and the inverse relation ?

Spherical Polar (r,θ,ϕ) : two obvious examples





Spherical Polar (r,θ,ϕ)



$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
$$

$r = [0, \infty]$

$\theta = [0, \pi]$

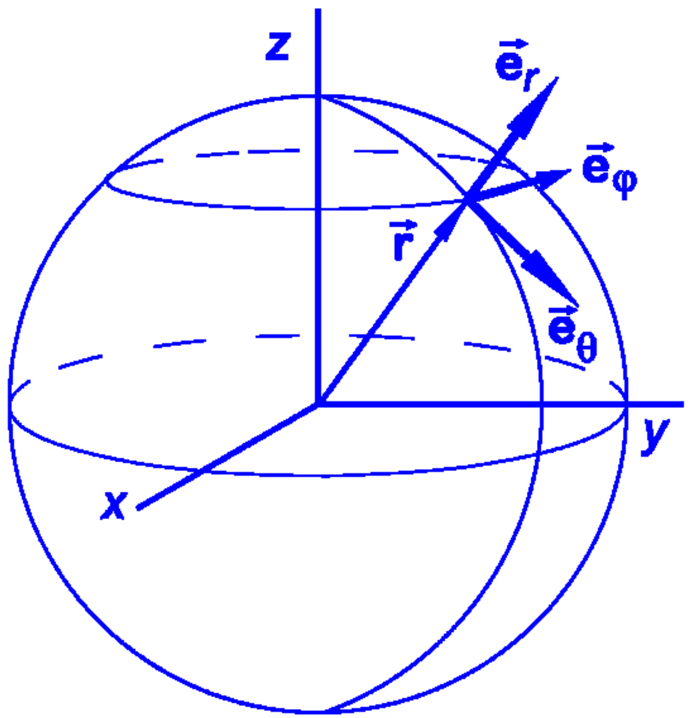
$\phi = [0, 2\pi]$

$$
\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \begin{pmatrix} \sin\theta\ \cos\phi & r\ \cos\theta\ \cos\phi & -r\ \sin\theta\ \sin\phi \\ \sin\theta\ \sin\phi & r\ \cos\theta\ \sin\phi & r\ \sin\theta\ \cos\phi \\ \cos\theta & -r\ \sin\theta & 0 \end{pmatrix} \begin{pmatrix} \delta r \\ \delta\theta \\ \delta\phi \end{pmatrix}
$$

Spherical Polar (r,θ,ϕ) : unit vectors, volume element, arc length

$$
\begin{aligned}
\boldsymbol{\epsilon}_r &= \sin\theta\cos\phi\ \boldsymbol{i} + \sin\theta\sin\phi\ \boldsymbol{j} + \cos\theta\ \boldsymbol{k}\\
\boldsymbol{\epsilon}_\theta &= \cos\theta\cos\phi\ \boldsymbol{i} + \cos\theta\sin\phi\ \boldsymbol{j} + -\sin\theta\ \boldsymbol{k}\\
\boldsymbol{\epsilon}_\phi &= -\sin\phi\ \boldsymbol{i} + \cos\phi\ \boldsymbol{j}
\end{aligned}
$$

Can you invert this set of equations? It is easy!



$$
\begin{aligned}
\delta \boldsymbol{r} &= \boldsymbol{\epsilon}_r\,\delta r + \boldsymbol{\epsilon}_\theta\, r\,\delta\theta + \boldsymbol{\epsilon}_\phi\, r\sin\theta\,\delta\phi\\
ds^2 &= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2
\end{aligned}
$$

$$
\begin{aligned}
dV &= \left|\ \boldsymbol{\epsilon}_r.(\boldsymbol{\epsilon}_\theta r)\times(\boldsymbol{\epsilon}_\phi r\sin\theta)\ \right|\ dr\ d\theta\ d\phi\\
&= r^2\sin\theta\ dr\ d\theta\ d\phi
\end{aligned}
$$

Spherical Polar (r,θ,ϕ) : The area element

$$If \quad r = constant \quad (surface \quad of \quad a \quad sphere) \quad \delta r = 0$$

$$\begin{aligned} dA &= \left| r\,\boldsymbol{\epsilon}_\theta \times r\,\sin\theta\,\boldsymbol{\epsilon}_\phi \right| d\theta\, d\phi \\ &= r^2 \sin\theta\, d\theta\, d\phi \end{aligned}$$

$$If \quad \theta = constant \quad \delta\theta = 0$$

$$\begin{aligned} dA &= \left| \boldsymbol{\epsilon}_r \times r\,\sin\theta\,\boldsymbol{\epsilon}_\phi \right| dr\, d\phi \\ &= r \sin\theta\, dr\, d\phi \end{aligned}$$

$$If \quad \phi = constant \quad (plane \quad polar \quad \text{in} \quad a \quad vertical \quad plane) \quad \delta\phi = 0$$

$$\begin{aligned} dA &= \left| \boldsymbol{\epsilon}_r \times r\,\boldsymbol{\epsilon}_\theta \right| dr\, d\theta \\ &= r\, dr\, d\theta \end{aligned}$$

Q:
Suppose you were confined on the surface of a sphere – but you were not told that. Would you be able to figure out?

Spherical Polar (r,θ,ϕ) : velocity & acceleration

We still need to express the derivatives $\left(\dot{\boldsymbol{\epsilon}}_r, \dot{\boldsymbol{\epsilon}}_\theta, \dot{\boldsymbol{\epsilon}}_\phi\right)$ in *terms of* $\left(\boldsymbol{\epsilon}_r, \boldsymbol{\epsilon}_\theta, \boldsymbol{\epsilon}_\phi\right)$

$$\begin{pmatrix} \dot{\boldsymbol{\epsilon}}_r \\ \dot{\boldsymbol{\epsilon}}_\theta \\ \dot{\boldsymbol{\epsilon}}_\phi \end{pmatrix} = \dot{\boldsymbol{M}}\, \boldsymbol{M}^T \begin{pmatrix} \boldsymbol{\epsilon}_r \\ \boldsymbol{\epsilon}_\theta \\ \boldsymbol{\epsilon}_\phi \end{pmatrix}$$

$$\boldsymbol{M} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \quad \boldsymbol{M}^T = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

$$\dot{\boldsymbol{M}} = \begin{pmatrix} \cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi} & \cos\theta\sin\phi\,\dot{\theta} + \sin\theta\cos\phi\,\dot{\phi} & -\sin\theta\,\dot{\theta} \\ -\sin\theta\cos\phi\,\dot{\theta} - \cos\theta\sin\phi\,\dot{\phi} & -\sin\theta\sin\phi\,\dot{\theta} + \cos\theta\cos\phi\,\dot{\phi} & -\cos\theta\,\dot{\theta} \\ -\cos\phi\,\dot{\phi} & -\sin\phi\,\dot{\phi} & 0 \end{pmatrix}$$

Spherical Polar (r,θ,ϕ) : velocity & acceleration

This appears very messy! But if you work through the matrix multiplication then:

$$\begin{pmatrix} \dot{\boldsymbol{\epsilon}}_r \\ \dot{\boldsymbol{\epsilon}}_\theta \\ \dot{\boldsymbol{\epsilon}}_\phi \end{pmatrix} = \dot{\boldsymbol{M}}\,\boldsymbol{M}^T \begin{pmatrix} \boldsymbol{\epsilon}_r \\ \boldsymbol{\epsilon}_\theta \\ \boldsymbol{\epsilon}_\phi \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \dot{\theta} & \sin\theta\,\dot{\phi} \\ -\dot{\theta} & 0 & \cos\theta\,\dot{\phi} \\ -\sin\theta\,\dot{\phi} & -\cos\theta\,\dot{\phi} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\epsilon}_r \\ \boldsymbol{\epsilon}_\theta \\ \boldsymbol{\epsilon}_\phi \end{pmatrix}$$

The result is remarkably simple.

Why are the diagonal terms zero? Can you see the physical implication?

Notice that the matrix connecting the two vectors is anti-symmetric.

This was also the case in the plane polar co-ordinates. But we didn't mention it there.

The problem for velocity and acceleration components can now be completed...

Spherical Polar (r,θ,φ) : The gradient

If we have a function $T(r,\theta,\phi)$ *then we want*

$$\delta T = \frac{\partial T}{\partial r}\delta r + \frac{\partial T}{\partial \theta}\delta\theta + \frac{\partial T}{\partial \phi}\delta\phi$$

$$= \boldsymbol{\nabla} T . \delta \boldsymbol{r}$$

since

$$\delta \boldsymbol{r} = \boldsymbol{\epsilon}_r \delta r + \boldsymbol{\epsilon}_\theta r \delta\theta + \boldsymbol{\epsilon}_\phi r \sin\theta \, \delta\phi$$

we must have

$$\boldsymbol{\nabla} T = \boldsymbol{\epsilon}_r \frac{\partial T}{\partial r} + \boldsymbol{\epsilon}_\theta \frac{1}{r}\frac{\partial T}{\partial \theta} + \boldsymbol{\epsilon}_\phi \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

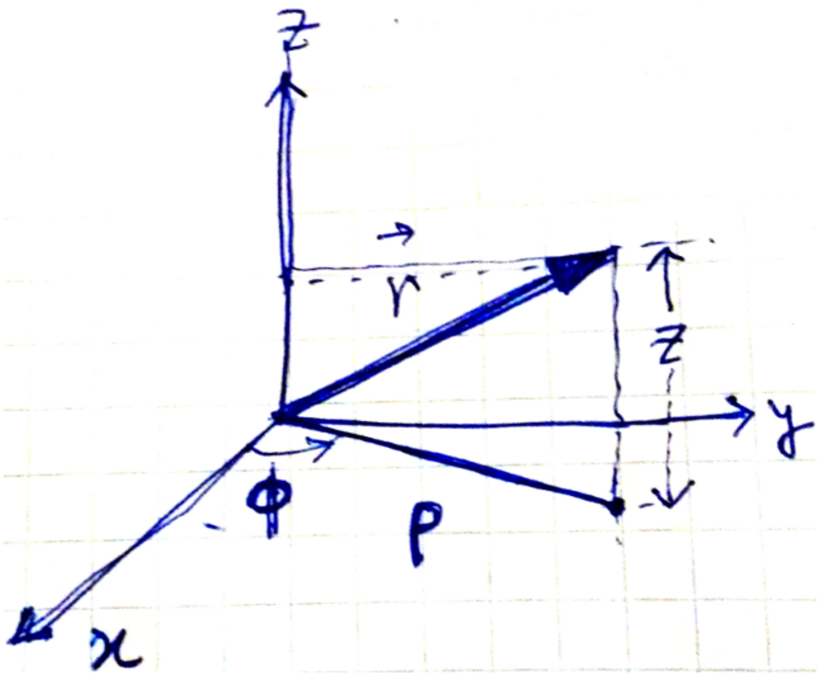
Spherical Polar (r,θ,ϕ) : velocity & acceleration

You should be able to show the following now:

$$
\begin{aligned}
\boldsymbol{v} &= \boldsymbol{\epsilon}_r\, \dot{r} + \boldsymbol{\epsilon}_\theta\, r\, \dot{\theta} + \boldsymbol{\epsilon}_\phi\, r \sin\theta\, \dot{\phi} \\
\boldsymbol{a} &= \boldsymbol{\epsilon}_r \left(\ddot{r} - r\,\dot{\theta}^2 - r\,\dot{\phi}^2 \sin^2\theta \right) + \\
&\quad \boldsymbol{\epsilon}_\theta \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} - r\,\dot{\phi}^2 \sin\theta\cos\theta \right) + \\
&\quad \boldsymbol{\epsilon}_\phi \left(r\,\ddot{\phi}\sin\theta + 2\,\dot{r}\,\dot{\phi}\sin\theta + 2r\,\dot{\theta}\,\dot{\phi}\cos\theta \right)
\end{aligned}
$$

We now have all the necessary bits to solve
dynamical problems in this co-ordinate

Cylindrical polar (ρ,θ,z) : unit vectors



$$
\begin{aligned}
x &= \rho\cos\phi \\
y &= \rho\sin\phi \\
z &= z
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\epsilon}_\rho &= \cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j} \\
\boldsymbol{\epsilon}_\phi &= -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j} \\
\boldsymbol{\epsilon}_z &= \boldsymbol{k}
\end{aligned}
$$

$$
\delta\boldsymbol{r} = \boldsymbol{\epsilon}_\rho\,\delta\rho + \boldsymbol{\epsilon}_\phi\,\rho\,\delta\phi + \boldsymbol{\epsilon}_z\,\delta z
$$

Wires, co-axial cables, Pipes etc.

Cylindrical polar (ρ,θ,z) : length, area and volume elements

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

$$dA = \rho\, d\rho\, d\phi \qquad z = constant$$

$$dA = \rho\, d\phi\, dz \qquad \rho = constant$$

$$dA = d\rho\, dz \qquad \phi = constant$$

volume

$$dV = \rho\, d\rho\, d\phi\, dz$$

gradient

$$\nabla = \boldsymbol{\epsilon}_\rho \frac{\partial}{\partial \rho} + \boldsymbol{\epsilon}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \boldsymbol{\epsilon}_z \frac{\partial}{\partial z}$$

Follow exactly the same process as we did for spherical polar...

Generic orthogonal curvilinear co-ordinates : unit vector, scale factor

Writing the basic information about orthogonal co-ordinates....

$$d\boldsymbol{r} = \boldsymbol{\epsilon_1} h_1 du_1 + \boldsymbol{\epsilon_2} h_2 du_2 + \boldsymbol{\epsilon_3} h_3 du_3$$

$$ds^2 = ?$$

$$dV = ?$$

A shorthand compact way of writing co-ordinates

$$d\boldsymbol{r} = \sum \boldsymbol{\epsilon_i} h_i d u_i$$

Summation convention :

REPEATED INDEX IMPLIES SUMMATION

$$d\boldsymbol{r} = \boldsymbol{\epsilon_i} h_i d u_i$$

Flux and circulation

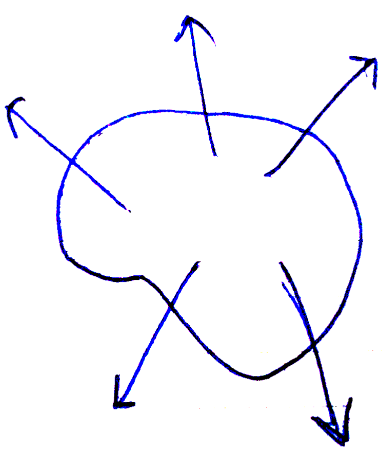
The volume of water flowing out through the SURFACE per unit time

$$\oint\!\!\!\oint \boldsymbol{v} . d\boldsymbol{S}$$

The shape of the surface can be arbitrary

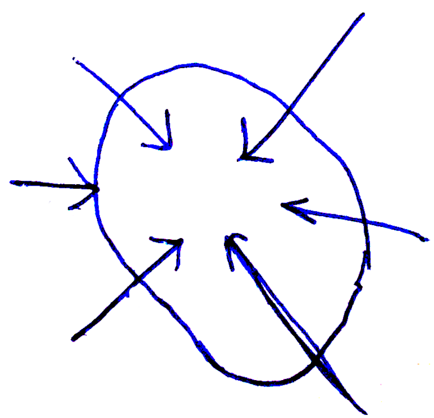
dS points OUTWARD

This has a unique meaning only if the surface is closed.



something flowing out

$$\oint\!\!\!\oint \boldsymbol{v} . d\boldsymbol{S} > 0$$



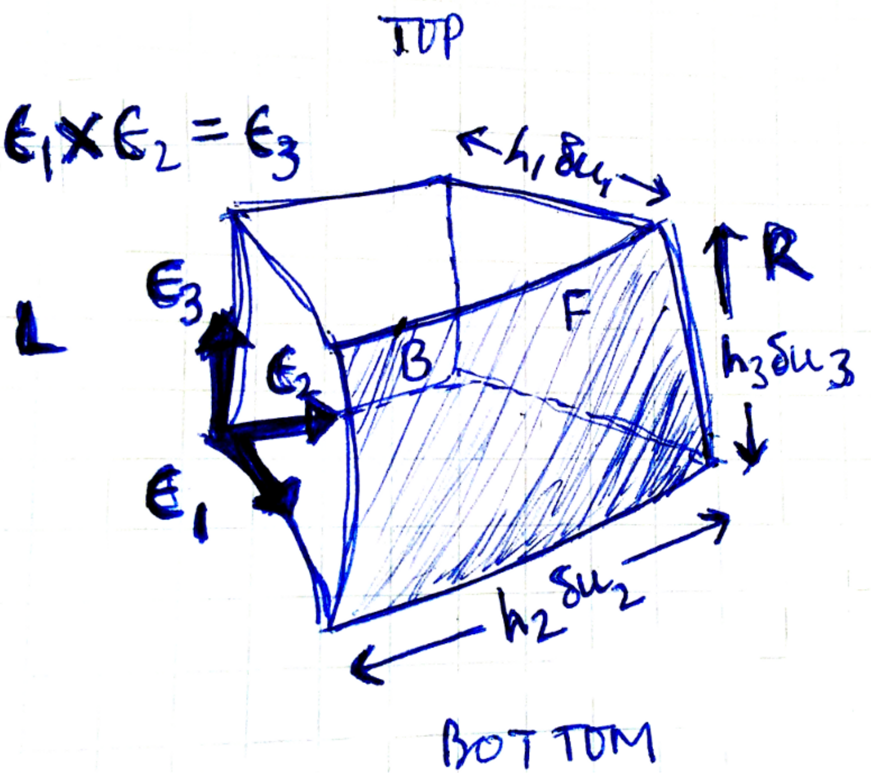
something flowing in

$$\oint\!\!\!\oint \boldsymbol{v} . d\boldsymbol{S} < 0$$

Consider a vector $\boldsymbol{F}$

Is it possible to have a function $\boldsymbol{X}(\boldsymbol{F})$ such that

$$X(\boldsymbol{F})\, dV = \boldsymbol{F} . d\boldsymbol{S}$$



Flux through BACK

$$f_B = -F_1 h_2 \delta u_2 h_3 \delta u_3$$

Flux through FRONT

$$f_F = F_1 h_2 \delta u_2 h_3 \delta u_3 + \frac{\partial}{\partial u_1}\left(F_1 h_2 \delta u_2 h_3 \delta u_3\right) \delta u_1$$

$$f_B + f_F = \left[\frac{\partial}{\partial u_1}\left(F_1 h_2 h_3\right)\right] \delta u_1 \delta u_2 \delta u_3$$

!! BE VERY CLEAR ABOUT THE SIGN OF EACH QUANTITY !!

Flux and circulation

The LEFT + RIGHT pair gives

$$f_L + f_R = \left[\frac{\partial}{\partial u_2}\left(F_2 h_1 h_3\right)\right]\delta u_1 \delta u_2 \delta u_3$$

The BOTTOM + TOP pair gives

$$f_{Bottom} + f_{Top} = \left[\frac{\partial}{\partial u_3}\left(F_3 h_1 h_2\right)\right]\delta u_1 \delta u_2 \delta u_3$$

$$f_{TOTAL} = \left[\frac{\partial}{\partial u_1}\left(F_1 h_2 h_3\right) + \frac{\partial}{\partial u_2}\left(F_2 h_1 h_3\right) + \frac{\partial}{\partial u_3}\left(F_3 h_1 h_2\right)\right]\delta u_1 \delta u_2 \delta u_3$$

$$\frac{\boldsymbol{F}.\delta \boldsymbol{S}}{\delta V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(F_1 h_2 h_3\right) + \frac{\partial}{\partial u_2}\left(F_2 h_3 h_1\right) + \frac{\partial}{\partial u_3}\left(F_3 h_1 h_2\right)\right]$$

Now break a finite volume into small volume elements

Flux from neighbouring walls of two infinitesimal volume elements will cancel

Only faces which form the part of the boundary of the volume will not cancel

This function is called DIVERGENCE, denoted by $\nabla . \boldsymbol{F}$

$$\oiiint \nabla . \boldsymbol{F}\, dV = \oiint \boldsymbol{F} . d\boldsymbol{S}$$

Called Gauss's theorem

"divergence" should convey a visual picture of the Vector field.... What is it?

How should a vector field look around points of stable/unstable equilibrium ?

Divergence of a vector is a scalar quantity

In Cartesian:

$$\nabla . \boldsymbol{F} = \boldsymbol{i}\frac{\partial F_x}{\partial x} + \boldsymbol{i}\frac{\partial F_y}{\partial y} + \boldsymbol{k}\frac{\partial F_z}{\partial z}$$

In Spherical polar:

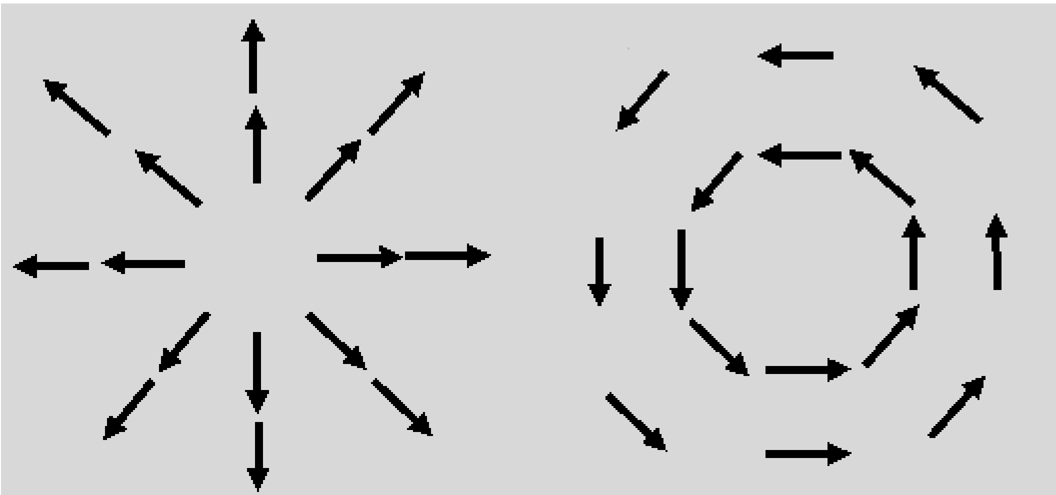
$$\nabla . \boldsymbol{F} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta\, F_r\right) + \frac{\partial}{\partial \theta}\left(r \sin\theta\, F_\theta\right) + \frac{\partial}{\partial \phi}\left(r F_\phi\right)\right]$$

In cylindrical polar

$$\nabla . \boldsymbol{F} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho F_\rho\right) + \frac{\partial}{\partial \phi}\left(F_\phi\right) + \frac{\partial}{\partial z}\left(\rho F_z\right)\right]$$

Divergence and continuity equation....

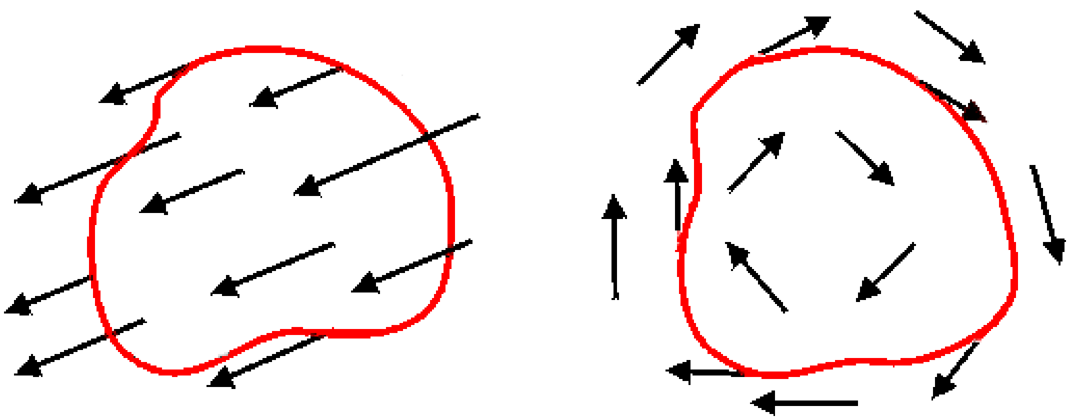
Flux and circulation



$$\oiint \boldsymbol{F} \cdot d\boldsymbol{S}$$

identifies a distinctive field pattern.

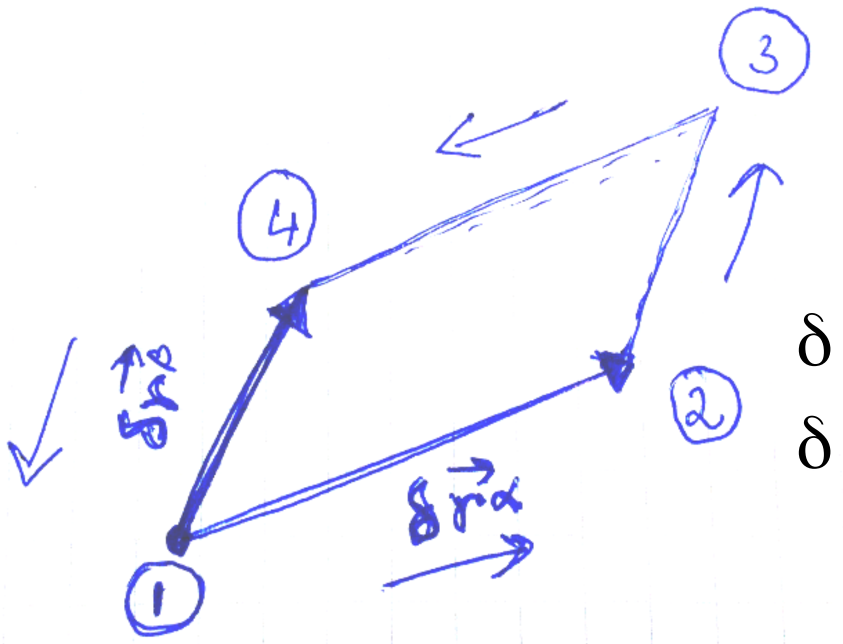
Another possible one is a circulating pattern.



When will $\oint \boldsymbol{F} \cdot d\boldsymbol{l}$ *be nonzero ?*

Flux and circulation

Consider two arbitray infinitesimal displacements



$$\delta \boldsymbol{r}^{\boldsymbol{\alpha}} = \boldsymbol{\epsilon}_{\mathbf{1}} h_1 \delta u_1^{\alpha} + \boldsymbol{\epsilon}_{\mathbf{2}} h_2 \delta u_2^{\alpha} + \boldsymbol{\epsilon}_{\mathbf{3}} h_3 \delta u_3^{\alpha}$$

$$\delta \boldsymbol{r}^{\boldsymbol{\beta}} = \boldsymbol{\epsilon}_{\mathbf{1}} h_1 \delta u_1^{\beta} + \boldsymbol{\epsilon}_{\mathbf{2}} h_2 \delta u_2^{\beta} + \boldsymbol{\epsilon}_{\mathbf{3}} h_3 \delta u_3^{\beta}$$

The vector field is $\boldsymbol{F}$.
Is it possible to have a function $\boldsymbol{X}(\boldsymbol{F})$ such that

$$X(\boldsymbol{F}).\delta \boldsymbol{S} = \sum_{\substack{peri-\\meter}} \boldsymbol{F}.\delta \boldsymbol{l}$$

If possible then this function will connect some characteristics of inside points with the boundary

Flux and circulation

$$d\boldsymbol{S} = \delta \boldsymbol{r}^{\alpha} \times \delta \boldsymbol{r}^{\beta} = \begin{vmatrix} \boldsymbol{\epsilon}_1 & \boldsymbol{\epsilon}_2 & \boldsymbol{\epsilon}_3 \\ h_1 \delta u_1^{\alpha} & h_2 \delta u_2^{\alpha} & h_3 \delta u_3^{\alpha} \\ h_1 \delta u_1^{\beta} & h_2 \delta u_2^{\beta} & h_3 \delta u_3^{\beta} \end{vmatrix}$$

$$\begin{aligned} X(\boldsymbol{F}).d\boldsymbol{S} = \; & X_1 h_2 h_3 [\delta u_2^{\alpha} \delta u_3^{\beta} - \delta u_3^{\alpha} \delta u_2^{\beta}] \\ & - X_2 h_1 h_3 [\delta u_1^{\alpha} \delta u_3^{\beta} - \delta u_3^{\alpha} \delta u_1^{\beta}] \\ & + X_3 h_1 h_2 [\delta u_1^{\alpha} \delta u_2^{\beta} - \delta u_2^{\alpha} \delta u_1^{\beta}] \end{aligned}$$

Try writing RHS in this form and compare.
The co-efficients of the arbitrary displacments must agree

!! BE VERY CLEAR ABOUT THE SIGN OF EACH QUANTITY !!

Flux and circulation

Consider the pair of paths $(1 \to 2)$ and $(3 \to 4)$

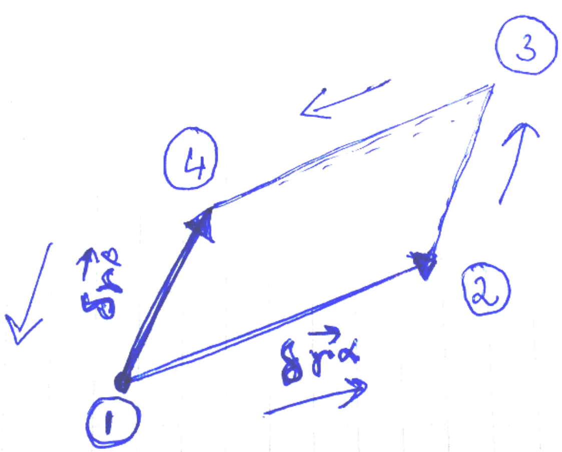
$$\boldsymbol{F}.\delta \boldsymbol{l}\big|_{1\to 2} = F_1 h_1 \delta u_1^\alpha + F_2 h_2 \delta u_2^\alpha + F_3 h_3 \delta u_3^\alpha$$

$$\boldsymbol{F}.\delta \boldsymbol{l}\big|_{3\to 4} = \left[F_i h_i + (\nabla F_i h_i).\delta \boldsymbol{r}^\beta\right](-\delta u_i^\alpha) \qquad (i=1,2,3)$$

Write contributions from $\boldsymbol{F}.\delta \boldsymbol{l}\big|_{2\to 3}$ & $\boldsymbol{F}.\delta \boldsymbol{l}\big|_{4\to 1}$ similarly.

Full path gives:

$$\left(\nabla \boldsymbol{F}.\delta r^\beta\right).\delta r^\alpha - \left(\nabla \boldsymbol{F}.\delta r^\alpha\right).\delta r^\beta$$



$$= \sum_{k,i}\left[\frac{1}{h_k}\frac{\partial F_i h_i}{\partial u_k}\delta u_i^\beta\right] h_k \delta u_k^\alpha$$

$$- \sum_{k,i}\left[\frac{1}{h_k}\frac{\partial F_i h_i}{\partial u_k}\delta u_i^\alpha\right] h_k \delta u_k^\beta$$

$$= \sum_{k,i}\left[\frac{\partial F_i h_i}{\partial u_k} - \frac{\partial F_k h_k}{\partial u_i}\right]\delta u_i^\beta \delta u_k^\alpha$$

!! BE VERY CLEAR ABOUT THE SIGN OF EACH QUANTITY !!

Flux and circulation

Now compare the co-efficient of $\delta u_2^\alpha \delta u_3^\beta - \delta u_3^\alpha \delta u_2^\beta$

We need to put $i=3, k=2$ and then $i=2, k=3$

this gives $X_1 h_2 h_3 = \left[\frac{\partial F_3 h_3}{\partial u_2} - \frac{\partial F_2 h_2}{\partial u_3} \right]$

So $$X(\boldsymbol{F}) = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \boldsymbol{\epsilon_1} & h_2 \boldsymbol{\epsilon_2} & h_3 \boldsymbol{\epsilon_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \equiv \begin{cases} \nabla \times \boldsymbol{F} \\ curl\ \boldsymbol{F} \\ rot\ \boldsymbol{F} \end{cases}$$

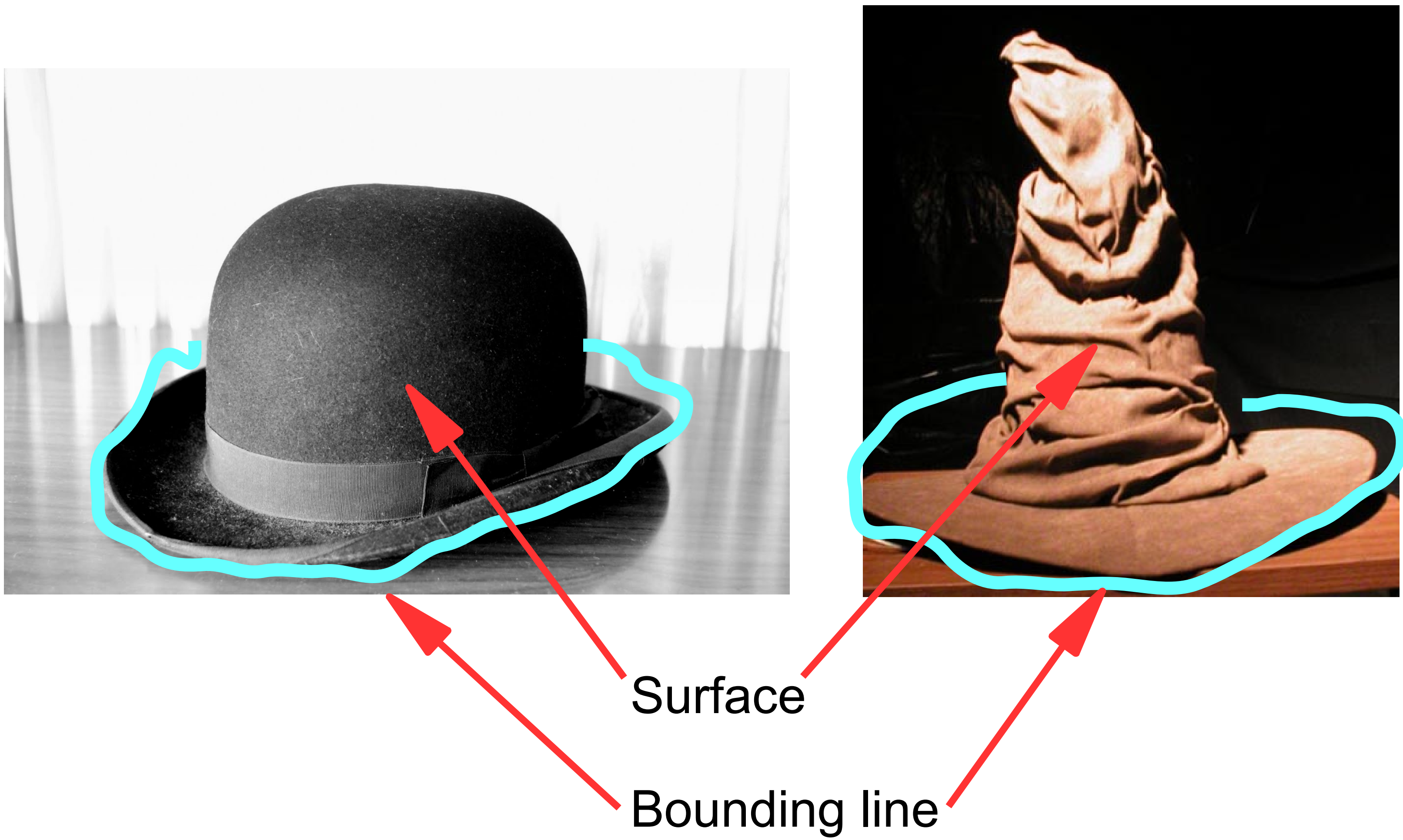
We have $\iint \nabla \times \boldsymbol{F} . d\boldsymbol{S} = \oint \boldsymbol{F} . d\boldsymbol{l}$ $\quad (called\ Stoke's\ theorem)$

Now break a finite surface into small area elements

Line integral from neighbouring perimeters of two infinitesimal area elements will cancel

Only line segments which form the part of the perimeter will not cancel

Flux and circulation : Which surface?



Any surface with the same bounding edge will work.

Curl F over any closed surface should be zero. WHY?

Divergence of a curl = ?

Curl of a gradient = ?

Multiple vector products : ε–δ notation (Levi-Civita)

Write the dot product as $\boldsymbol{A}.\boldsymbol{B} = \delta_{ij} A_i B_j \quad where \;\; \delta_{ij} = \begin{cases} 1 & if \;\; i=j \\ 0 & if \;\; i \neq j \end{cases}$

Write the cross product as

$$\boldsymbol{A} \times \boldsymbol{B}|_i = \epsilon_{ijk} A_j B_k \quad where \;\; \epsilon_{ijk} = \begin{cases} 1 & if \;\; ijk \;\; is \;\; an \;\; even \;\; permutation \;\; of \;\; 123 \\ -1 & if \;\; ijk \;\; is \;\; an \;\; odd \;\; permutation \;\; of \;\; 123 \\ 0 & otherwise \end{cases}$$

Convince yourself that $\epsilon_{ijk} = \boldsymbol{\epsilon}_i . \boldsymbol{\epsilon}_j \times \boldsymbol{\epsilon}_k$

This works with operators also: $with \;\; x_i \;\; for \;\; x, y, z$

$$\nabla . \boldsymbol{A} = \frac{\partial A_i}{\partial x_i}$$

$$\nabla \times \boldsymbol{A}\Big|_i = \epsilon_{ijk} \frac{\partial A_j}{\partial x_k}$$

Notice how the summation convention on repeated indices have been used.

Q: How does it help?

Multiple vector products : ε–δ notation (Levi-Civita)

Consider a vector triple product $\boldsymbol{A}\times\left(\boldsymbol{B}\times\boldsymbol{C}\right)$

$$
\begin{aligned}
\boldsymbol{A}\times\left(\boldsymbol{B}\times\boldsymbol{C}\right)\Big|_i &= \epsilon_{ijk}A_j\left(\boldsymbol{B}\times\boldsymbol{C}\right)_k \\
&= \epsilon_{ijk}A_j\left(\epsilon_{kpq}B_pC_q\right) \\
&= \epsilon_{kij}\epsilon_{kpq}A_jB_pC_q
\end{aligned}
$$

What to do with a product like $\epsilon_{ijk}\epsilon_{kpq}$?

This is either -1 or 0 or 1

WITHOUT summation,

we can write for a generic product term: $\left(why\,?\right)$

kpq itself is some permutation of the sequence lmn, otherwise the product will vanish.

$$
\begin{aligned}
\epsilon_{lmn}\epsilon_{kpq} = \;& \delta_{lk}\delta_{mp}\delta_{nq} + \delta_{lp}\delta_{mq}\delta_{nk} + \delta_{lq}\delta_{mk}\delta_{np} \\
& -\delta_{lk}\delta_{mq}\delta_{np} - \delta_{lp}\delta_{mk}\delta_{nq} - \delta_{lq}\delta_{mp}\delta_{nk}
\end{aligned}
$$

odd/even permutations

kpq	$+$
kqp	$-$
pkq	$-$
pqk	$+$
qkp	$+$
qpk	$-$

Multiple vector products : $\varepsilon-\delta$ notation (Levi-Civita)

$$
\begin{aligned}
\epsilon_{kij}\epsilon_{kpq} &= \delta_{kk}\delta_{ip}\delta_{jq} + \delta_{kp}\delta_{iq}\delta_{jk} + \delta_{kq}\delta_{ik}\delta_{jp} \\
&\quad -\delta_{kk}\delta_{iq}\delta_{jp} - \delta_{kp}\delta_{ik}\delta_{jq} - \delta_{kq}\delta_{ip}\delta_{jk} \\
&= \delta_{kk}\left(\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}\right)+\delta_{kp}\left(\delta_{iq}\delta_{jk}-\delta_{ik}\delta_{jq}\right)+\delta_{kq}\left(\delta_{ik}\delta_{jp}-\delta_{ip}\delta_{jk}\right) \\
&= 3\left(\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}\right)+\left(\delta_{iq}\delta_{jp}-\delta_{ip}\delta_{jq}\right)+\left(\delta_{iq}\delta_{jp}-\delta_{ip}\delta_{jq}\right) \\
&= \delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}
\end{aligned}
$$

sum over k

$$\textit{Using the last result} \left(\textit{with} \quad i=p\right)$$

$$
\begin{aligned}
\epsilon_{kij}\epsilon_{kiq} &= \delta_{ii}\delta_{jq}-\delta_{iq}\delta_{ji} \\
&= 3\delta_{jq}-\delta_{jq} \\
&= 2\delta_{jq}
\end{aligned}
$$

sum over k and i

$$\textit{Using the last result} \left(\textit{with} \quad j=q\right)$$

$$
\begin{aligned}
\epsilon_{kij}\epsilon_{kij} &= 2\delta_{jj} \\
&= 6
\end{aligned}
$$

sum over k,i and j

Successive summation over indices.

The first sum is most frequently encountered. It allows you to write a cross product in terms of dot product like terms

Multiple vector products : ε–δ notation (Levi-Civita)

Should be able to prove all of these easily....
(list from the last page of Griffith's book)

TRIPLE PRODUCTS

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

PRODUCT RULES

(3) $\nabla(fg)=f(\nabla g)+g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A})=f(\nabla\cdot\mathbf{A})+\mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A})=f(\nabla\times\mathbf{A})-\mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}+\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})$

SECOND DERIVATIVES

(9) $\nabla\cdot(\nabla\times\mathbf{A})=0$

(10) $\nabla\times(\nabla f)=0$

(11) $\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$

Co-ordinate systems, transformations etc : few problems

You are given the set of co-ordinate transformation equations, say $(r, \theta, \phi) \rightarrow (x, y, z)$
What is the value of the following determinant $|J|$*, where*

$$J = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} & \dfrac{\partial z}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial z}{\partial \theta} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial z}{\partial \phi} \end{pmatrix}$$

What is the physical significance of the result?
If the transformation was non-orthogonal,
would the same significance still hold?
Can you make use of J*, for calculating the inverse* partial derivatives
$\dfrac{\partial \theta}{\partial x}$ *as a function of* (r, θ, ϕ)*?*

When is a vector field uniquely specified?

$$\text{Suppose} \quad \boldsymbol{A} = \nabla \times \boldsymbol{F} \text{ is specified.}$$
$$\text{Can we solve for } \boldsymbol{F}\text{ ?}$$

ϕ(x,y,z) is any scalar field

$$\text{NO.} \quad \boldsymbol{F}' = \boldsymbol{F} + \nabla \phi \quad \text{will give same result.}$$

$$\text{Suppose} \quad \psi = \nabla . \boldsymbol{F} \text{ is specified.....}$$

B(x,y,z) is any vector field

$$\text{NO.} \quad \boldsymbol{F}' = \boldsymbol{F} + \nabla \times \boldsymbol{B} \quad \text{will give same result.}$$

$$\text{If} \quad \nabla \times \boldsymbol{F} \quad \text{AND} \quad \nabla . \boldsymbol{F} \text{ are given.}$$
$$\text{Then } \boldsymbol{F} \text{ can be solved for,}$$
$$\text{provided boundary conditions are also given.}$$

HELMHOLTZ'S THEOREM

$$\text{Maxwell's equations do precisely this.}$$

Gauss's law: Flux of Electric field through a closed surface

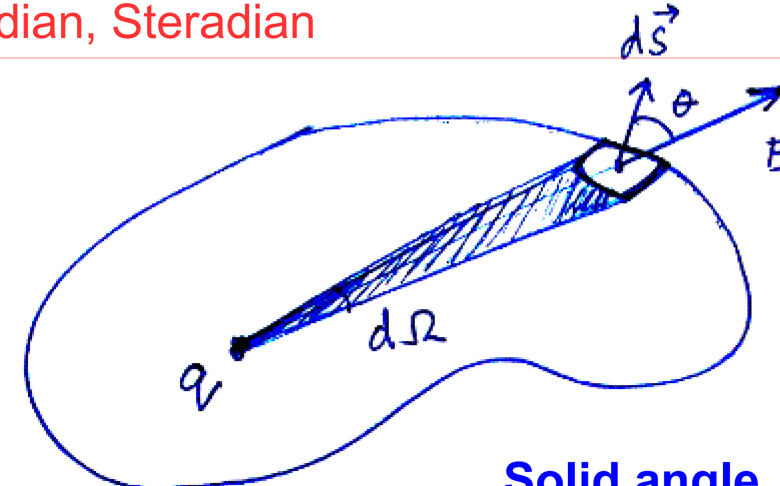
Revise the idea of planer angle and solid angle
Closed loop, Closed surface
Radian, Steradian

$$\int_{surface} \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{q}{4\pi\epsilon_0} \int_{surface} \frac{\hat{\boldsymbol{r}} \cdot d\boldsymbol{S}}{r^2}$$

$$= \frac{q}{4\pi\epsilon_0} \int_{surface} \frac{|d\boldsymbol{S}|\cos\theta}{r^2}$$

$$= \frac{q}{4\pi\epsilon_0} \int_{surface} d\Omega$$

$$= \frac{q}{4\pi\epsilon_0} 4\pi = \frac{q}{\epsilon_0}$$

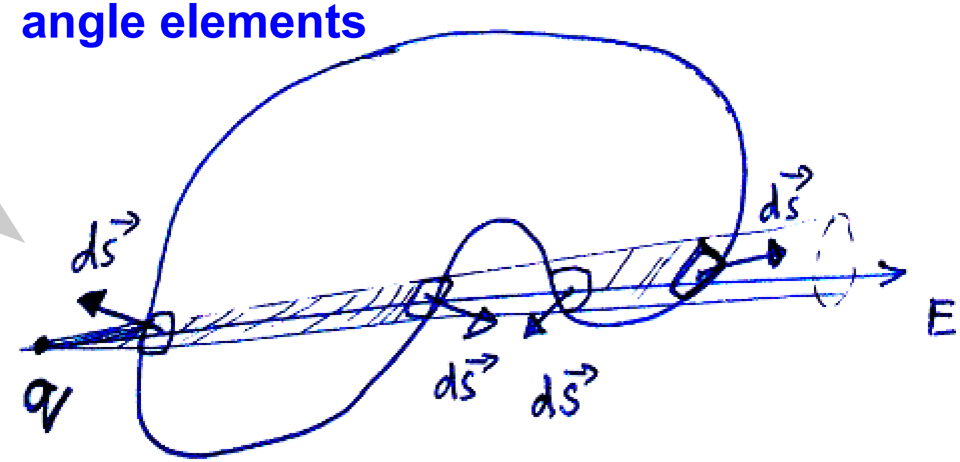


Solid angle

$$d\Omega = \frac{|ds|\cos\theta}{r^2}$$

If the point is located outside then the contributions exactly cancel

Notice the exact cancellation of solid angle elements



Use superposition principle
Add contribution from each charge

Gauss's law: Flux of Electric field through a closed surface

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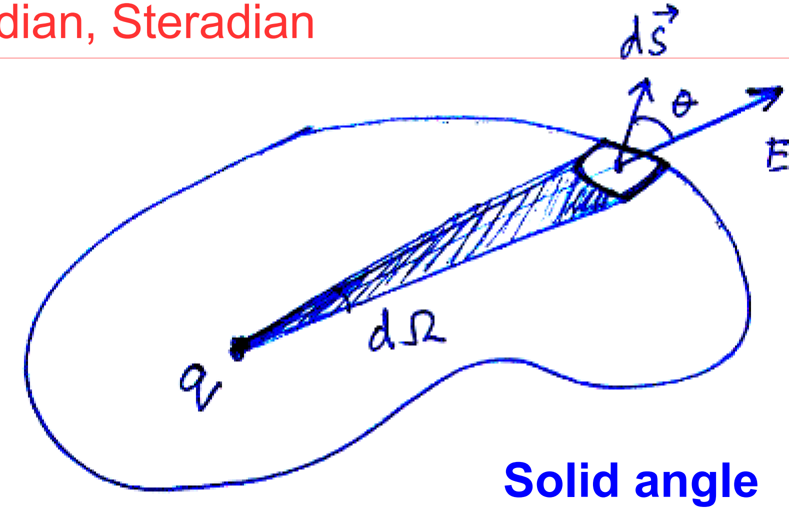

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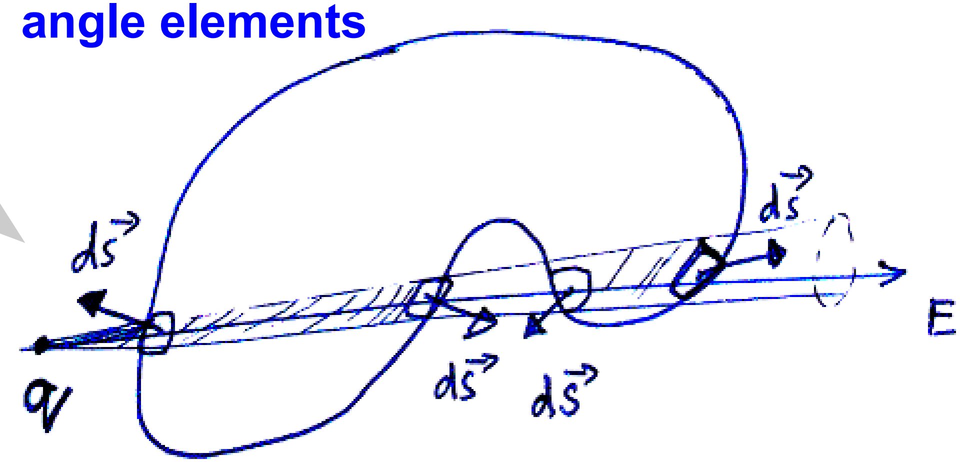


Use superposition principle

Add contribution from each charge

Gauss's law: Flux of Electric field through a closed surface

$$\int_{surface} \boldsymbol{E} \,.\, d\boldsymbol{S} = \int_{vol} \nabla \,.\, \boldsymbol{E} \, d\tau = \int_{vol} \frac{\rho(\boldsymbol{r})}{\epsilon_0} d\tau$$

Q: Why do we write This rather than leave Coulomb force law as it is?

$$So \quad \nabla \,.\, \boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0}$$

$$Also \quad \nabla^2 V = -\frac{\rho(\boldsymbol{r})}{\epsilon_0} \qquad since \quad \boldsymbol{E} = -\nabla V$$

Only when there is no time varying magnetic field. Same as saying Curl **E** = 0

This form allows one to use the symmetry of a problem more easily (e.g. sphere, infinite sheet, wire etc.)

Is valid even when charges are in motion.

Q: What is the problem with moving charges and Coulomb's law?

Fun question: If the world was 2-dimensional what would Coulomb's law be like? (Don't take it too seriously!)

How "exact" is the inverse square force law?

The cancellation of the $1/r^2$ came from two sources:
1. The geometrical growth of the area subtended by a small solid angle (geometry)
2. The nature of the coulomb force law (experimental observation)

If the force varied as $1/r^{2.0001}$, what observational consequence would it have?

Gauss's law: divergence of $1/r^2$ and the Dirac delta function

Do the following integration over a sphere

$$\int_{vol} \nabla \cdot \frac{\hat{\boldsymbol{r}}}{r^2}\, d\tau = \int_{surface} \frac{\hat{\boldsymbol{r}}}{r^2} \cdot d\boldsymbol{S} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta = 4\pi$$

But calculating the divergence explicitly

$$\nabla \cdot \frac{\hat{\boldsymbol{r}}}{r^2} = \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = 0$$

The inconsistency comes from the singularity at **r**=0

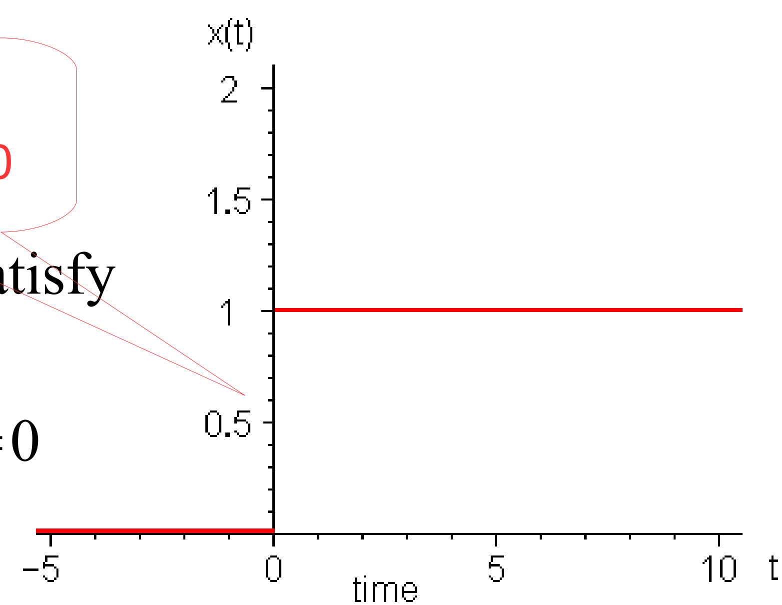
Consider a simpler example: the step function

$$x(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Integrable singularity at **r**=0

$\dfrac{dx}{dt} = ?$ looks like zero everywhere but must satisfy

$$\int_{0-|\epsilon|}^{0+|\epsilon|} \left(\frac{dx}{dt}\right) dt = x(|\epsilon|) - x(-|\epsilon|) = 1 \quad \forall \quad \epsilon \neq 0$$



Such integrable singularities are treated by *defining the* δ *function*

Gauss's law: divergence of 1/r² and the Dirac delta function

Such "functions" can only be defined by specifying their behaviour inside an integral. You cannot really plot such functions because they are inherently singular.

$$\int_a^b \delta(x-x_0) f(x)\, dx = \begin{cases} f(x_0), & \text{if } x_0 \text{ is within the limits} \\ 0 & \text{otherwise} \end{cases}$$

Visualise this as a huge spike at $x = x_0$ only,

Gets higher but narrower keeping the area under it, same.

Picks out the value of any $f(x)$ at the spike

Several other ways to define $\delta(x)$ *as a limit*

For our purpose, we will need to use

$$\boxed{\nabla \cdot \frac{\hat{\boldsymbol{r}}}{r^2} = 4\pi\,\delta(\boldsymbol{r})}$$

Fourier & Cauchy had introduced such "functions" Before.

In physics texts It is generally associated with Dirac

Gauss's law: divergence of 1/r² and the Dirac delta function

$$Some\ other\ \ integral\ representations\ of\ \delta\ \ function$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{ip(x-x_0)} = \delta(x-x_0)$$

Frequently used in Quantum mechanics

$$\lim_{a\to 0}\ \frac{1}{a\sqrt{\pi}}\, e^{-x^2/a^2} = \delta(x)$$

$$Try\ proving...\left(hint: change\ of\ variables\right)$$

$$\delta(-x) = \delta(x) \qquad \alpha\ \ is\ \ any\ \ constant$$

$$\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$$

The wikipedia article is excellent! Read it.

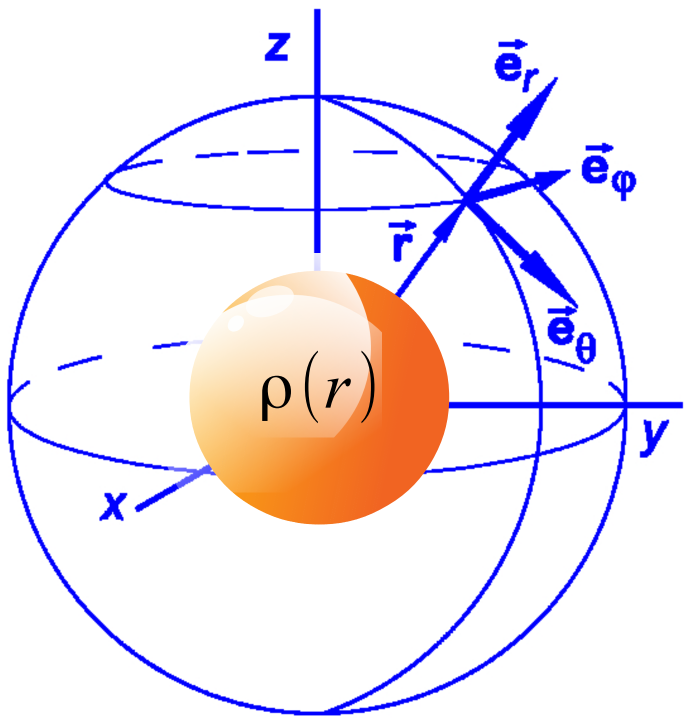
Electric field in some simple situations (symmetry + Gauss's Law)

No charge distribution is really infinite!

Very close to the surface/
Objects with very large aspect ratio

infinite approximation is useful

Spherically symmeteric charge distribution



$E_\theta = 0$ and $E_\phi = 0$. *Why?*

rotate about z-axis

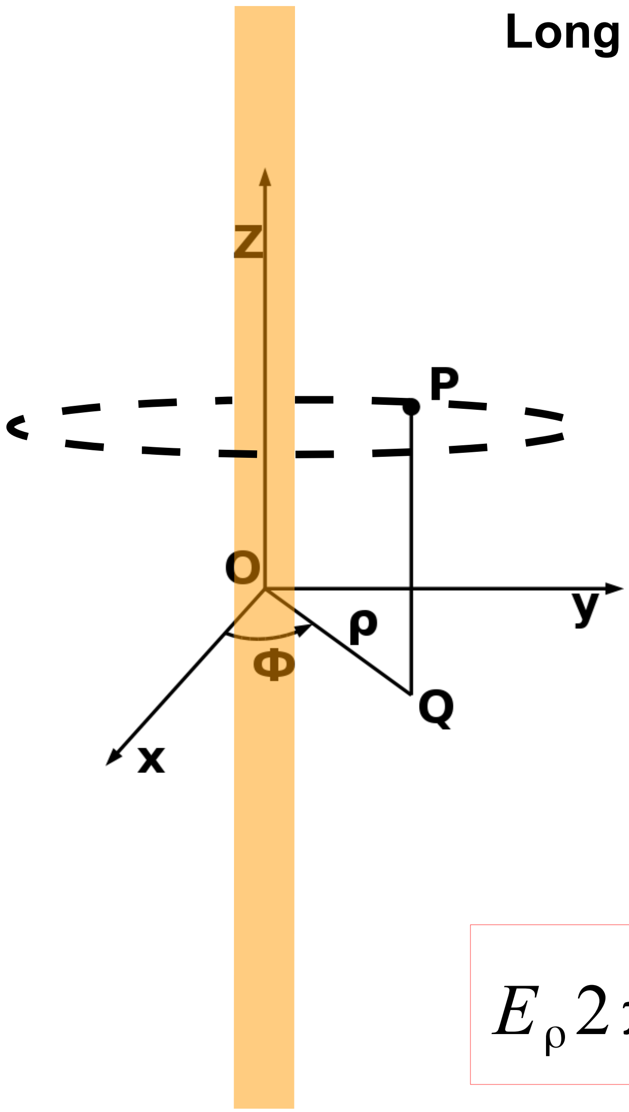
The field at the tip of $\boldsymbol{r}$, *must be same at all points visited by* $\boldsymbol{r}$

$$\oint \boldsymbol{E} \,.\, \boldsymbol{dl} = 0 \quad \rightarrow E_\phi = 0$$

Rotate about x-axis : show $E_\theta = 0$

$$\boxed{E_r 4\pi R^2 = \frac{1}{\epsilon_0}\int_0^R \rho(r) 4\pi r^2 dr}$$

Electric field in some simple situations (symmetry + Gauss's Law)



Long narrow wire type charge distribution

$E_\phi = 0$ and $E_z = 0$. *Why ?*

rotate about z-axis

The field at the tip of $\boldsymbol{P}$, *must be same at all points visited by* $\boldsymbol{P}$

$\oint \boldsymbol{E} \cdot \boldsymbol{dl} = 0 \quad \rightarrow E_\phi = 0$

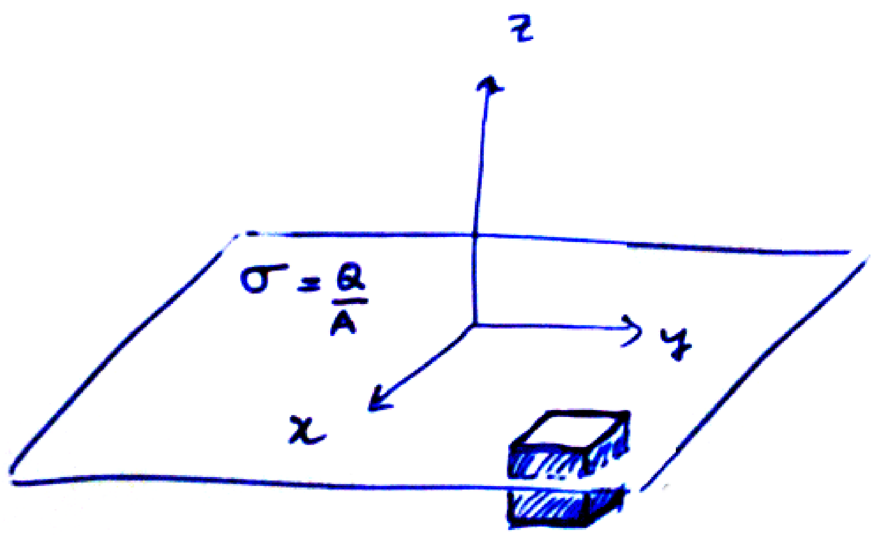
flip about z-axis : show $E_z = 0$

There is nothing to *chose z* from *-z*

$$E_\rho 2\pi\rho = \frac{1}{\epsilon_0}\lambda \qquad \lambda : \textit{charge per unit length}$$

Electric field in some simple situations (symmetry + Gauss's Law)

Infinite sheet of charge



Give a symmetry argument to show that E(z) = -E(-z) must hold.
Flip the "sheet" switching "topside" and "bottom-side". What should happen?

Why cannot there be a $E_{\parallel}$ *component ?*

Rotate the sheet about any point
Translate sheet by any in plane vector
Field cannot change.
Not possible if there is finite component.

$$E_z = \begin{cases} \dfrac{\sigma}{2\epsilon_0} & : z > 0 \\ -\dfrac{\sigma}{2\epsilon_0} & : z < 0 \end{cases}$$

Easy to extend to a sheet with finite thickness....work it out.

Superpose the field of two parallel plates, calculate capacitance

Some extra charge is placed in a conductor : why does it go to the surface?

$$\nabla . \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

Ohm's Law

$$\frac{1}{\sigma}\nabla . \boldsymbol{J} = \frac{\rho}{\epsilon_0}$$

Continuity equation

$$-\frac{\partial \rho}{\partial t} = \frac{\sigma}{\epsilon_0}\rho$$

$$\boxed{\rho = \rho(0) e^{-\frac{\sigma}{\epsilon_0}t}}$$

Div **J** = 0

steady current flow (as in a wire)
OR
J=0, **E**=0 (pure electrostatics)

In either case the excess charge is NOT in bulk

But charge is conserved.
Where does the excess charge go?
From "bulk“ to surface.

How fast?

For good metals like Cu, Ag, Au..

$$\sigma \sim 10^7 - 10^8\, S/m$$

$$\frac{\sigma}{\epsilon_0} \sim 10^{20}$$

$$time\ constant \sim 10^{-20}\ sec$$

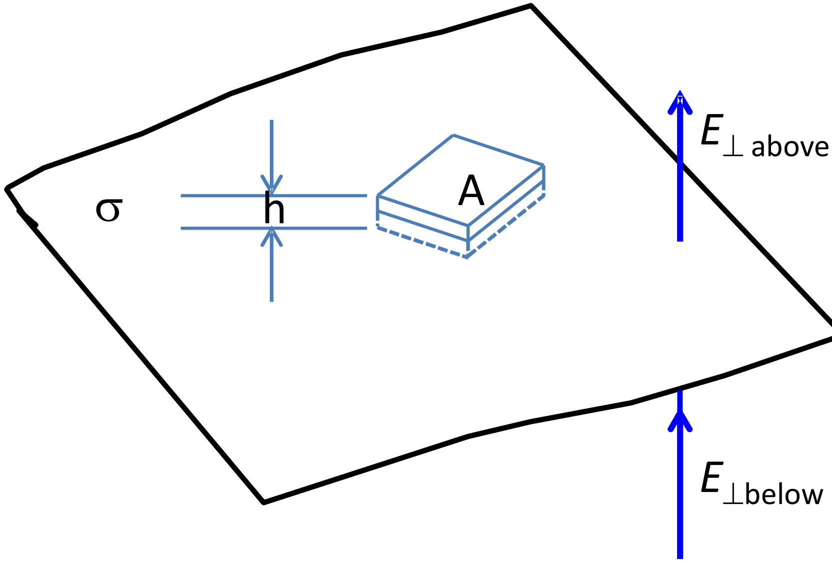
Boundary conditions and other characteristics of metals/conductors

You should now be able to justify the following :

- If no current is flowing in the conductor then E=0 inside
- All the excess charge resides on the surface of the conductor, even if there is a current flow.
- The conductor is an equipotential if J=0 (pure electrostatics)
- The electric field is normal to the surface (gradient is perpendicular to an equipotential)
- The surface charge density on a metal depends on the local radius of curvature.
- The electric field is strongest just outside sharp pointy edges.

Two spheres of radii R, r (R >>r) are in contact so at the same potential. Which one has a larger ***E*** *just outside? Why?*
Think of a cone as made of a series of successively smaller spheres...where is the electric field strongest?

Generic Electrostatic Boundary conditions (normal and tangential components)

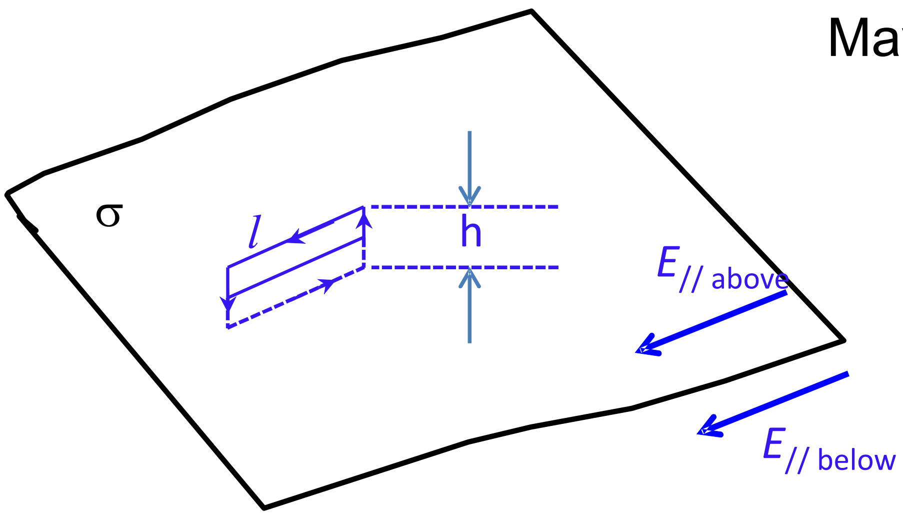


$$\oint\!\!\oint \boldsymbol{E}\,.\,d\boldsymbol{S} = \frac{Q_{enc}}{\epsilon_0}$$

$$E_{\perp above} - E_{\perp below} = \frac{\sigma}{\epsilon_0}$$

Normal component may be discontinuous

Surfaces may be finite.
The charge density may vary from place to place.
Surface is not necessarily equipotential
May be conducting or non-conducting



$$\oint \boldsymbol{E}\,.\,d\boldsymbol{l} = 0$$

$$E_{\| above} - E_{\| below} = 0$$

Tangnetial component is continuous,
But not necessarily zero.

But electrostatic potential V is always continuous at a surface.
Reason: The discontinuity in E is finite.
So $V_2 - V_1$ = -E.dl will go to zero as dl goes to zero

Solution of Laplace's equation: Average value theorem : statement

A scalar function $V(\boldsymbol{r})$ satisfies $\nabla^2 V = 0$

Consider a sphere of radius R : integrate $\nabla^2 V$ over the volume

$$\int_{vol} \boldsymbol{\nabla}.(\boldsymbol{\nabla} V)\, d\tau = \int_{surface} \boldsymbol{\nabla} V . d\boldsymbol{S}$$

Gradient in spherical polar

$$= \int [\boldsymbol{\epsilon}_r \frac{\partial V}{\partial r} + \boldsymbol{\epsilon}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} + \boldsymbol{\epsilon}_\phi \frac{1}{r \sin\theta}\frac{\partial V}{\partial \phi}] . d\boldsymbol{S}$$

$$= \int \frac{\partial V}{\partial r} R^2 \sin\theta\, d\theta\, d\phi$$

Only the radial component survives because d**S** points radially outwards

$$0 = R^2 \frac{\partial}{\partial r} \int_{surface} V(r, \theta, \phi) \sin\theta\, d\theta\, d\phi$$

The average value $\langle V(\theta, \phi) \rangle_r$ over a sphere is independent of r.

In the limit $r \to 0$, we must have $\langle V \rangle = V(0)$

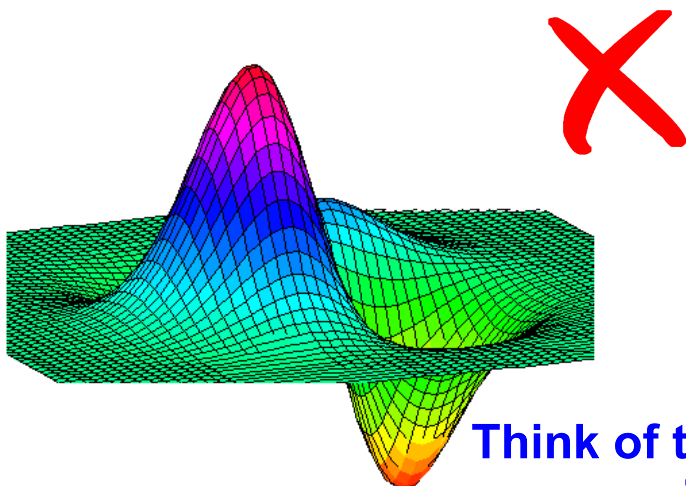
So average value over a spherical surface = value at the center

Prove the 2D (using plane polar) and 1d cases as exercise.

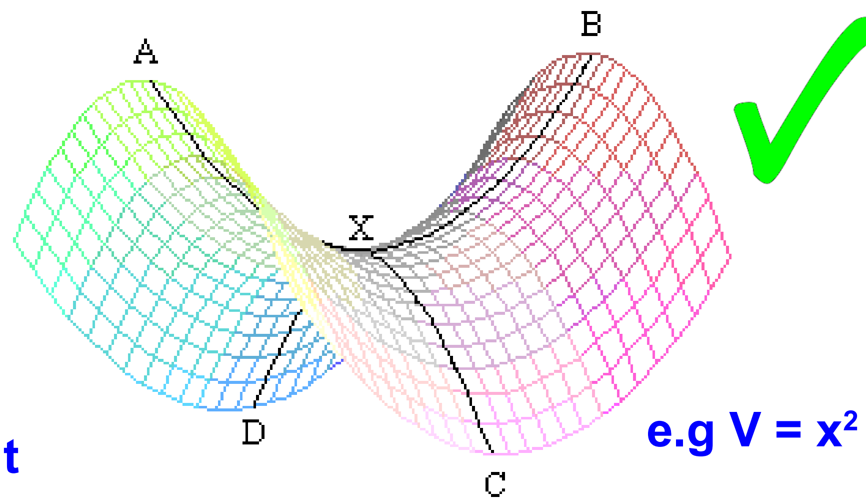
Solution of Laplace's equation: Average value theorem : consequences

There are no maxima or minima of V in a region where $\nabla^2 V = 0$

But there can be saddle points



Think of the gradient
near maxima/minima



e.g $V = x^2 - y^2$

No stable equilibrium possible in purely electrostatic field $(Earnshaw)$
All extremal values must occur at the boundary

$V = const$ on ALL points on ALL boundaries $\Rightarrow V$ is constant everywhere

UNIQUENESS: There is only one possible solution of $\nabla^2 V = -\frac{\rho}{\epsilon_0}$
consistent with a given boundary condition

Solution of Laplace's equation: Uniqueness theorem

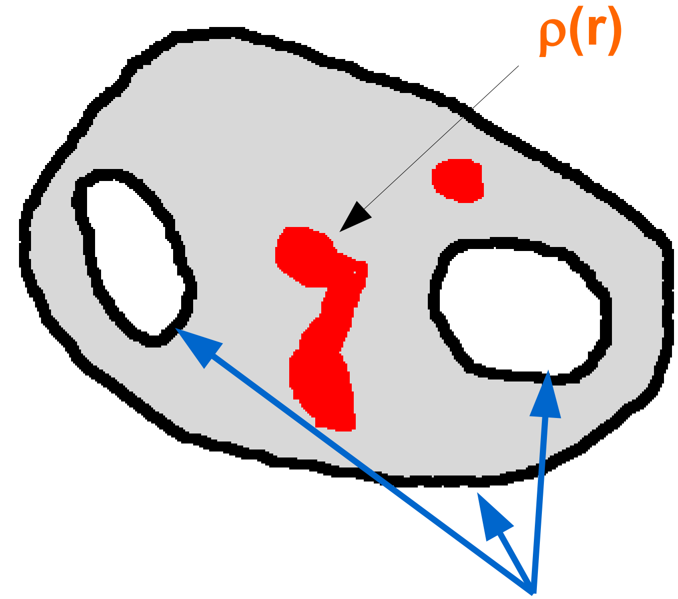
Two functions V_1 and V_2 satisfy $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

With the same boundary conditions

Let $\psi = V_1 - V_2$ Then

$\nabla^2 \psi = 0$ and $\psi = 0$ on ALL boundaries

Implies $\psi = 0$ everywhere



Another way to prove this:

consider the vector function: $\psi \nabla \psi$

$$\int_{vol} \nabla . (\psi \nabla \psi) d\tau = \int_{surface} \psi \nabla \psi . d\boldsymbol{S}$$

If a "guess" satisfies the boundary condition then that MUST be the solution

$$\int_{vol} \left[\psi \nabla^2 \psi + |\nabla \psi|^2 \right] d\tau = 0$$

$$\int_{vol} |\nabla \psi|^2 d\tau = 0$$

Possible only if ψ =constant=0 everywhere

Why is a metal cavity a "shield" ?

Arbitrary charges are outside the cavity (Q1...Qn)

Charges will be induced in the wall of the cavity.

But the wall remains an equipotential.

Inside the cavity V=0 is one possible solution satisfying the boundary conditions.

THAT IS THE UNIQUE SOLUTION.

What if the wall is not fixed at V=0 (i.e. floating)?

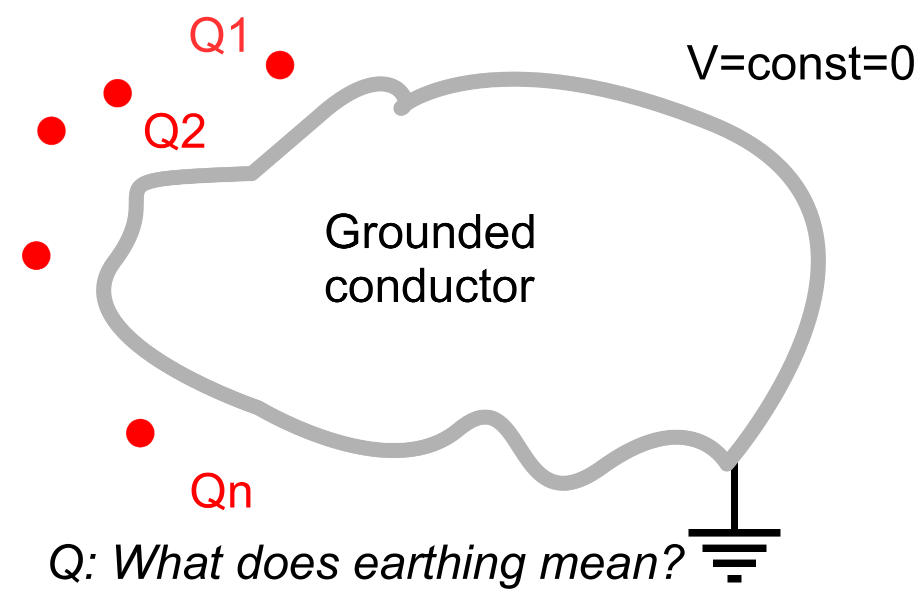
V=constant is still correct, but the constant will depend on the charge distribution outside.

If charges are placed inside?

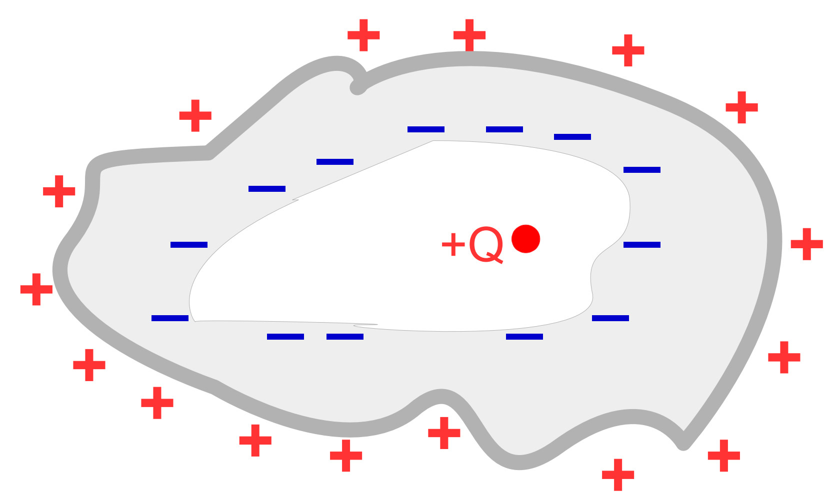
$$\nabla^2 V = -\frac{\rho_{in}}{\epsilon_0}$$

$V = 0$ (boundary condition)

irrespective of ρ_{out}

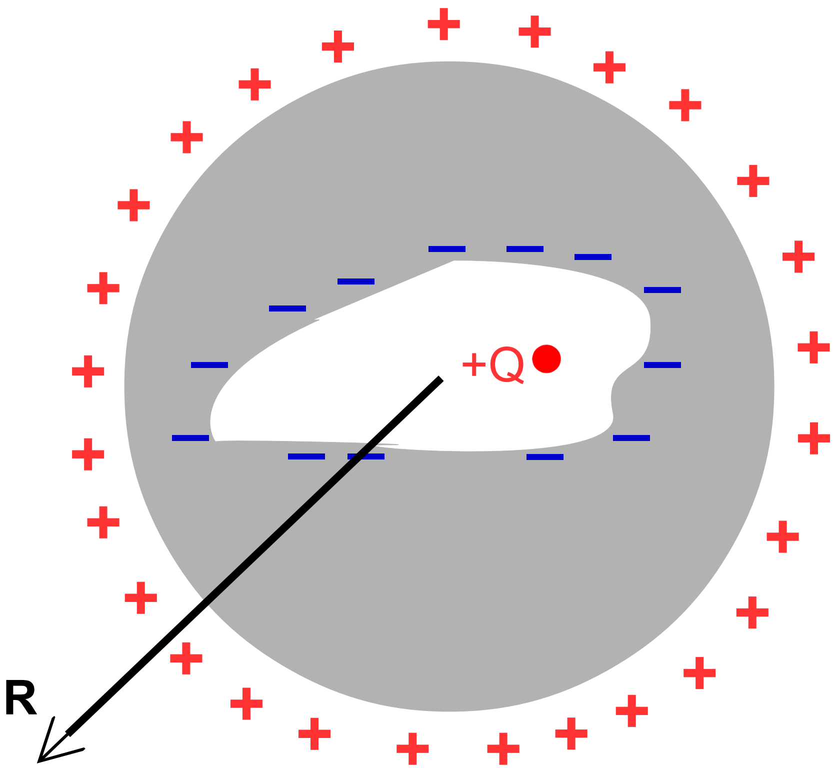


Q: What does earthing mean? Why is the potential of the earth fixed?



Floating conductor
Equal amounts +Q and -Q on inner and outer surfaces.

Why is a metal cavity a "shield" ?



Floating conductor
Equal amounts +Q and -Q on inner and outer surfaces.

Special case: irregular cavity inside a sphere.

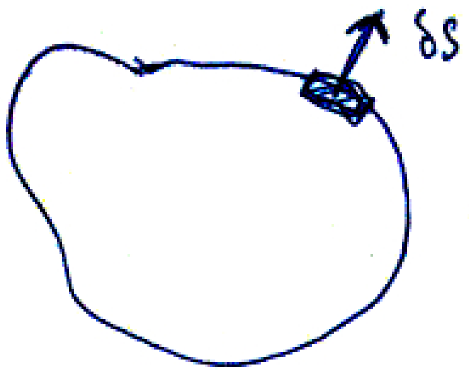
The charge density $\sigma(\theta,\phi)$ *is uniform*

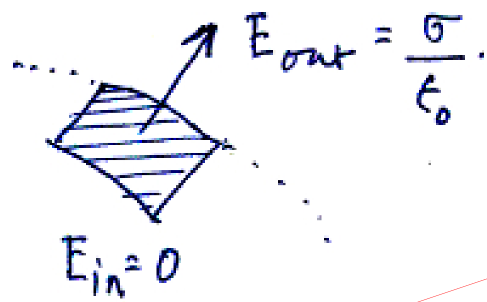
Surface is equipotential and E=0 inside. Use the boundary condition on Normal component of E and local charge density to prove the result.

$$E(R)=\frac{Q}{4\pi\epsilon_0 R^2}$$

Irrespective of the location of Q inside the cavity.

Electrostatic pressure on a conducting shell/ charged bubble



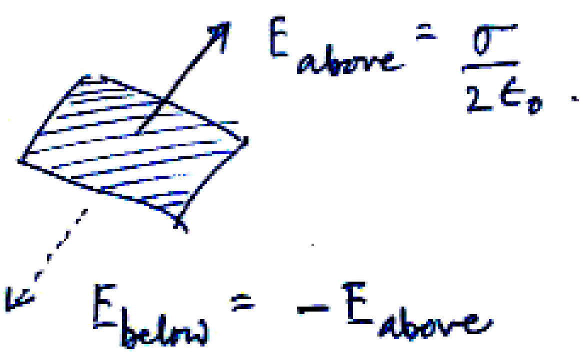


V=Const. | $E_{\parallel} = 0$.

If δs was treated in isolation

Force on the element dS is due to E field created by all the other charges.

$$\delta Q = \sigma \delta S \; creates$$
$$E_{\perp} = \frac{\sigma}{2\epsilon_0} \quad \text{above and below}$$
$$But \; E_{\perp} = \frac{\sigma}{\epsilon_0} \quad \text{(above)}$$
$$E_{\perp} = 0 \quad \text{(below/inside)}$$



The difference must be due to the field created by all the other charges.

The force on dS is
Charge in dS x Field due to all charges
NOT in dS

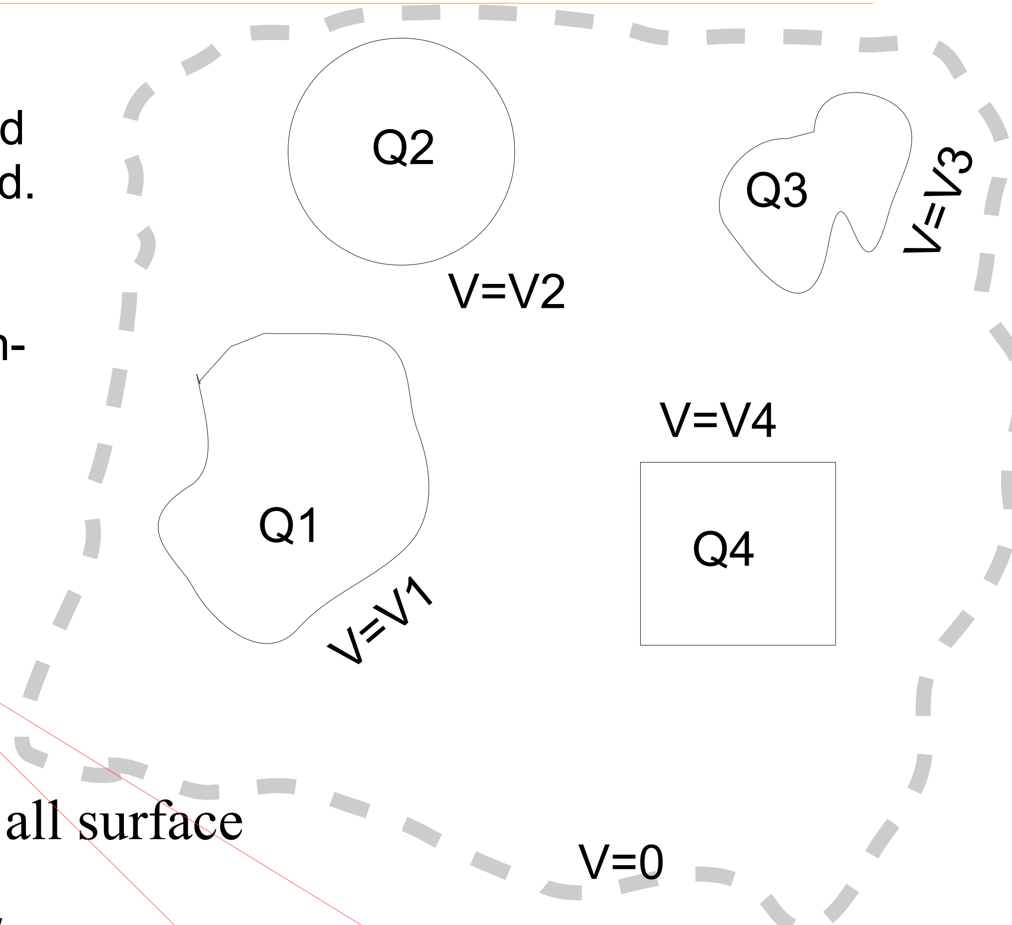
$$\delta F = (\sigma \delta S)\frac{\sigma}{2\epsilon_0} = \left(\frac{\sigma^2}{2\epsilon_0}\right)\delta S$$

Outward Pressure=Force/area
The conducting surface simplifies the calculation. For an arbitrary surface it is more complex.....

Electrostatics of conductors : uniqueness theorem 2 & capacitance

If the charge on ALL the **conductors** is specified then the potential V(x,y,z) is uniquely determined.

Notice that we are not specifying the charge distribution, only the total charge. That's the non-trivial content.



Suppose two distinct solutions exist

$U(x,y,z), V(x,y,z)$: *Both must satisfy*

$$\int_{surf\ i} (-\nabla U)\, d\tau = \int_{surf\ i} (-\nabla V)\, d\tau = Q_i$$

U and V must give equipotentials On each conducting surface, but we do not claim that they are the same constant to start with.

define $\psi = U - V$, & *integrate* $\psi \nabla \psi$ over all surface

$$\sum_i \int_{surf\ i} (-\nabla \psi).\, d\mathbf{S} = \int_{all\ vol} \left[\nabla^2 \psi + |\nabla \psi|^2\right] d\tau$$

= 0

$LHS = 0 : why?$

$$So \int_{all\ vol} |\nabla \psi|^2 d\tau = 0$$

$Hence \quad U - V = 0$

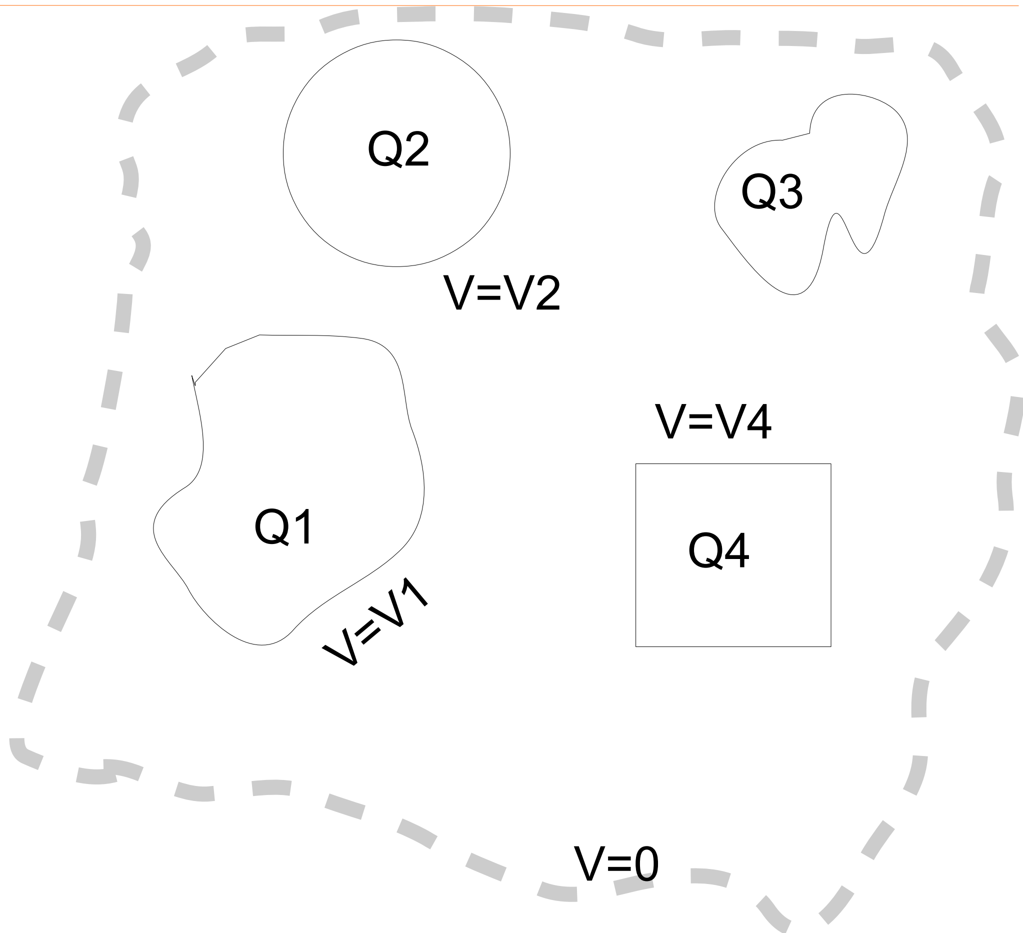
Electrostatics of conductors : uniqueness theorem 2 & capacitance

You are given the charge Q1, Q2....Qn
On each conductor.

You are not told what V1,V2....Vn are.

What is the most generic statement you can make?

$$Q_i = \sum_j C_{ij} V_j$$
$$C_{ij} = C_{ji}$$



The coefficients in this LINEAR relation are the formal definition of CAPACITANCE.

For a single object it reduces to the familiar relation : **Q=CV**

For an N-condcutor system the matrix is symmetric and can be inverted.

Try writing the energy of the system in matrix form as an exercise...

Where all do we come across Laplace's equation?

1. Fluid flow : Incompressible, "inviscid", "irrotational"

flow of "DRY water",
quite far from reality, still
useful as a starting point

$$(\rho = const. \quad \eta = 0) \;\Rightarrow\; \nabla . v = 0$$

$$If\ \boldsymbol{\nabla} \times \boldsymbol{v} = 0 \quad then \quad \boldsymbol{v} = \boldsymbol{\nabla}\phi \quad (velocity\ potential)$$

$$\nabla^2 \phi = 0$$

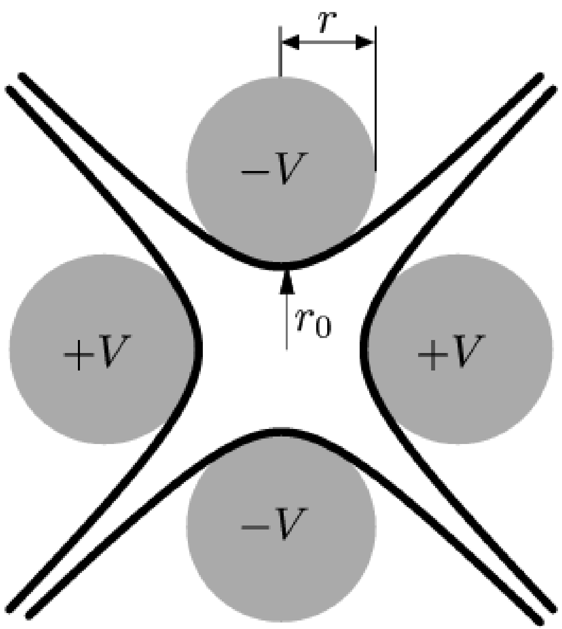
2. Heat conduction (Fourier), Diffusion equation (in steady state, time derivative =0)

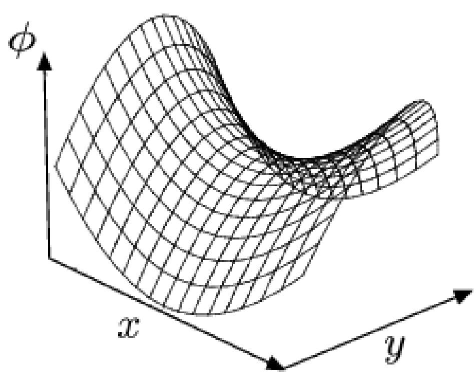
$$D\nabla^2\theta = \frac{\partial \theta}{\partial t}$$

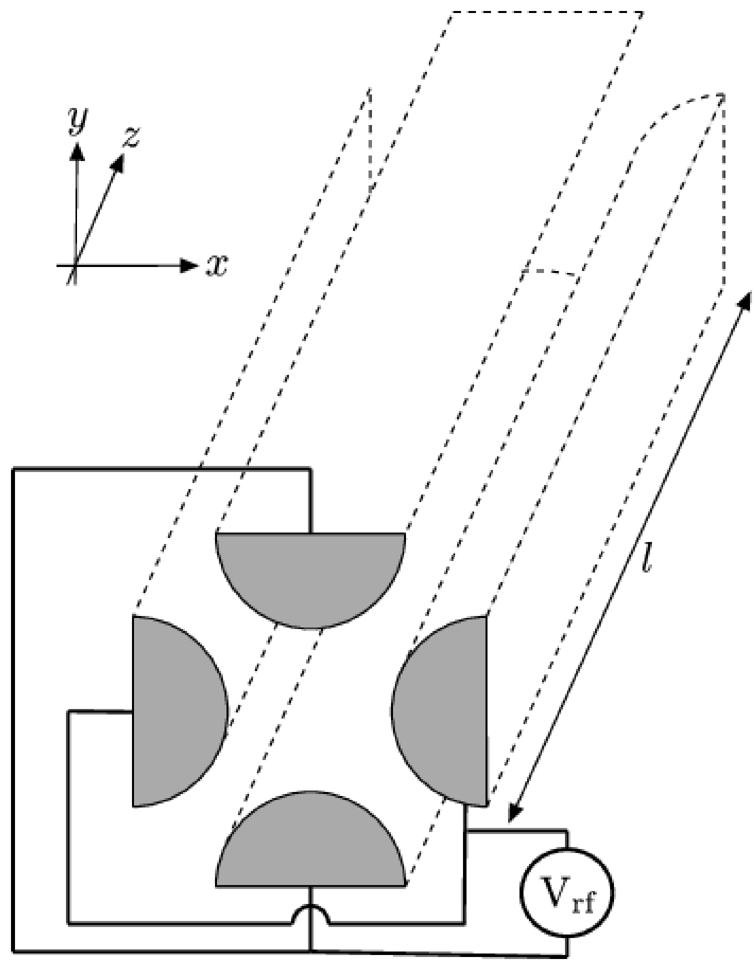
3. Electrostatic lensing :
Electron microscope, Ion trap, particle acceleration/beam steering, mass spectrometer

   Interesting differences from optical lensing:
      Charged nature of particles,
      Not possible to have focussing from all sides

Electrostatic "quadrupole" lens : an example

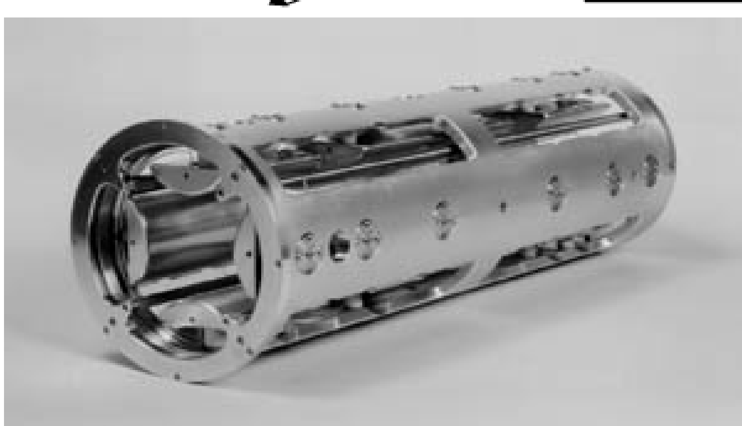






A schematic of an ideal hyperbolic electrostatic quadrupole (thick line) and the circular electrodes used to closely approximate the hyperbolic shape are shown on the top left. The saddle-shaped harmonic potential it creates is illustrated at the bottom left. The geometry formed from half-cylindrical rods and pair-wise applied a.c. potential (Vrf).

T. Brunner et al. *Nucl.Instrum.Meth.* **A676** (2012) 32-43



Notice how the problem becomes 2 dimensional
Translational symmetry of a long cylindrical geometry reduces many 3D problems to a 2D case.

Solving the Laplace equation : Image charge method

Problem: A charge distribution and some boundary conditions are given. The usual boundary conditions are fixed potentials over some surfaces.
Solve for V(r) in a certain region.

A "trick“ works for some (!! not all !!) problems.

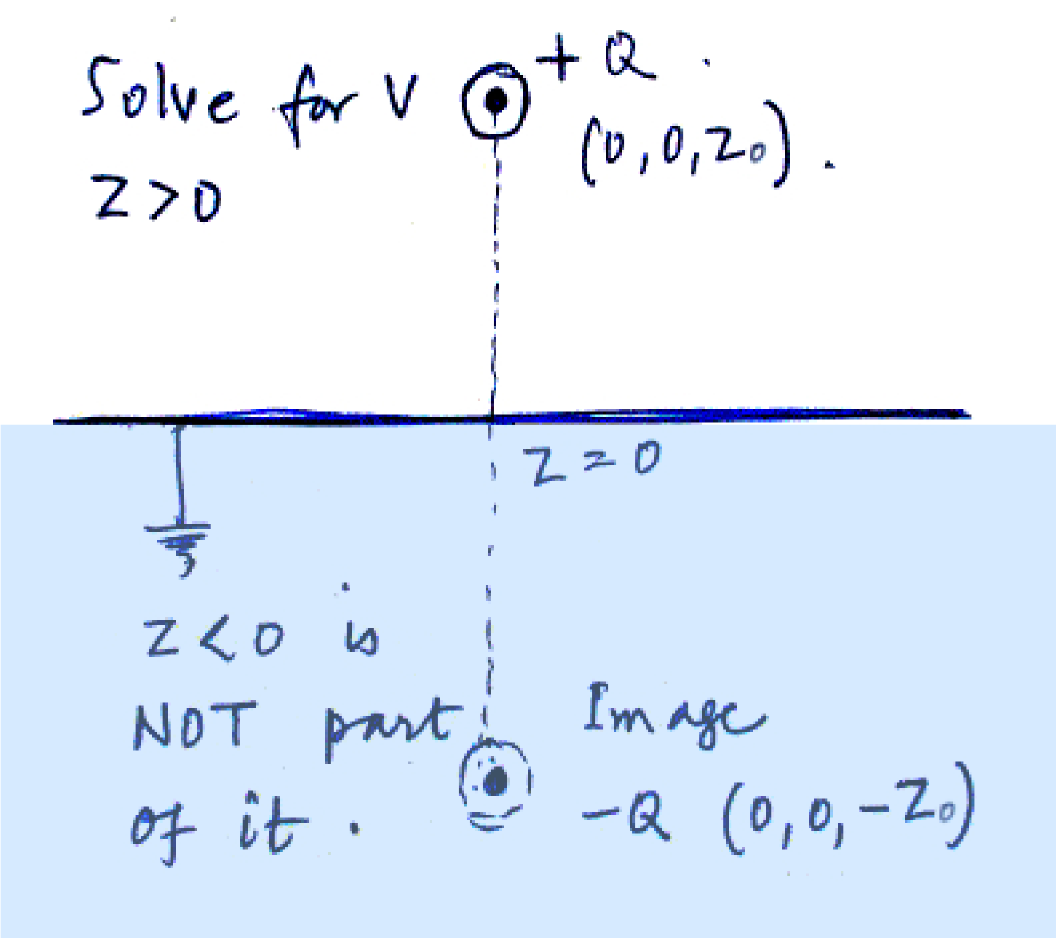
STEP 1: put some point charges in the regions NOT part of the region where you need to solve for the potential.

STEP 2: Try to arrange these external charges, so that the external + given charges together produce the desired potential at the boundaries. Forget all else!

STEP 3: Calculate the total potential in the certain (given) region using all the charges in the problem + external charges.

STEP 4: The total field/potential produced by the ALL the charges is the solution to the problem. The extra charges are called Image charges.

Image charge method : point charge near a conducting grounded plane



PROBLEM:
A point charge +Q is kept at $(0,0,z_0)$

z=0 is a grounded conducting sheet.

What is V(x,y,z) for z >0
Subject to boundary conditions:
V=0 at z=0
$V \longrightarrow 0$ as x,y,z $\longrightarrow \infty$

SOLUTION:

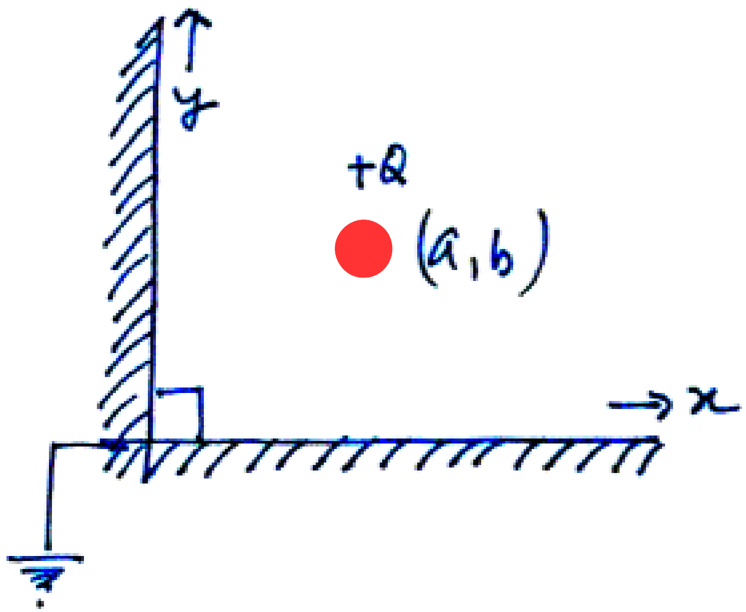
Put an extra charge -Q at $(0,0,-z_0)$

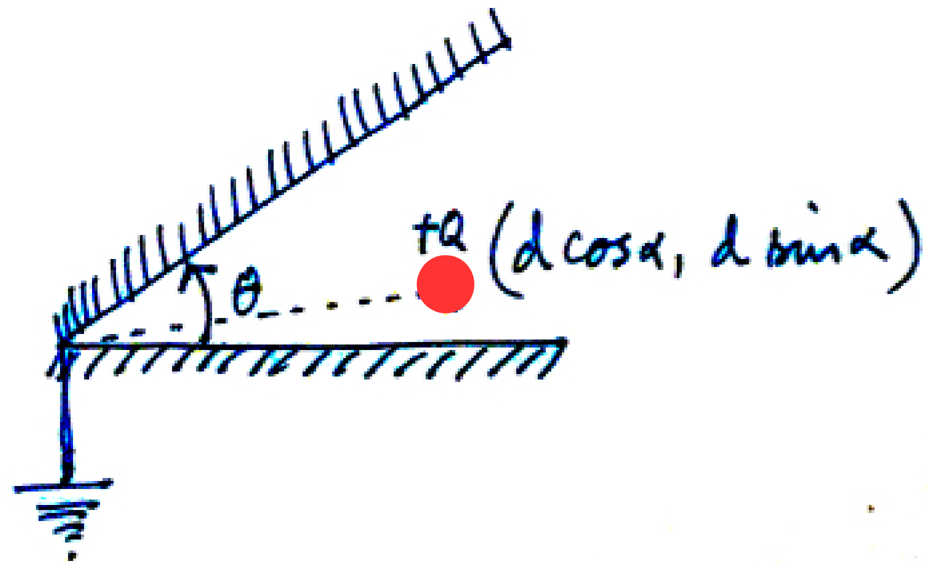
The potential due to both is

$$V(x,y,z)=\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-z_0)^2}}-\frac{1}{\sqrt{x^2+y^2+(z+z_0)^2}}\right]$$

For z=0, the two terms cancel giving V=0. This must be the solution (uniqueness).
We can now calculate the force between the charge +Q, induced surface charge at every point etc.

Image charge method : point charge near a conducting grounded plane





To solve this we need three image charges:

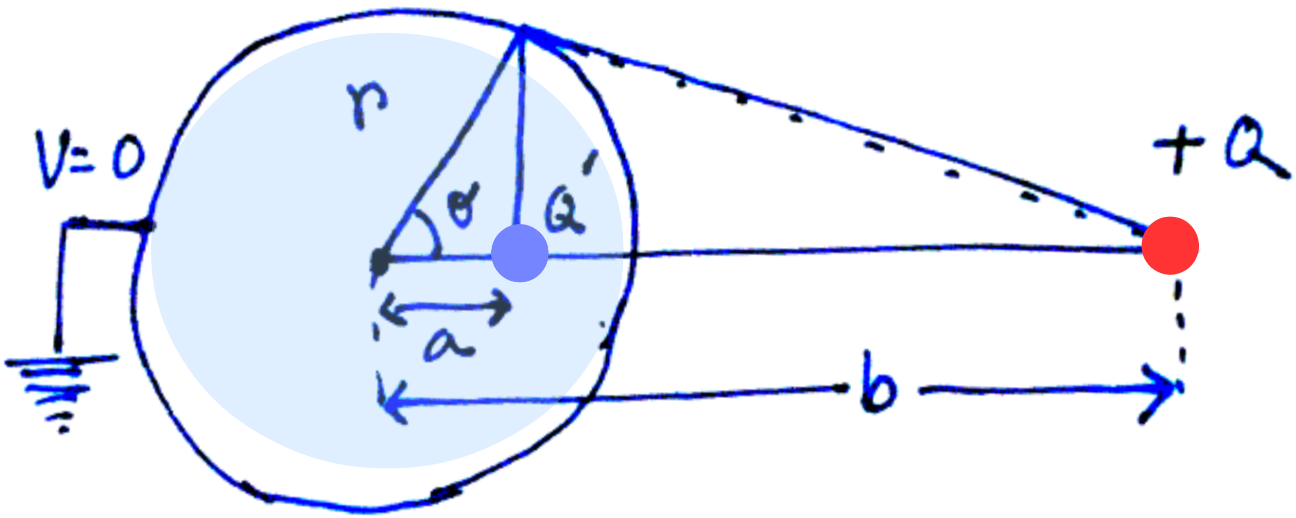
- Q at (a, -b)
+Q at (-a,-b)
- Q at (-a,b)

This problem can be solved with a finite number of images if

$$\theta \times integer = \pi$$

There is no generic method! It is a combination of guess and some calculation.....

Image charge method : point charge near a conducting grounded sphere



Solution wanted outside the sphere only

The distance of an arbitrary point on the surface from the charges Q and Q'

$$d'^2 = r^2 + a^2 - 2ar\cos\theta$$
$$d^2 = r^2 + b^2 - 2br\cos\theta$$

can we adjust a and Q' such that

$$\frac{Q}{d} - \frac{Q'}{d'} = 0 \quad \text{for all } \theta\,?$$

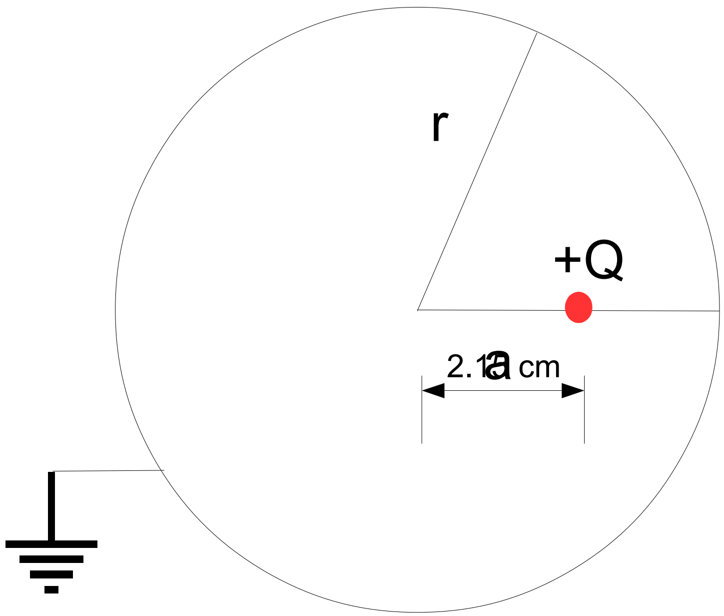
$$\frac{r^2 + a^2 - 2ar\cos\theta}{Q'^2} = \frac{r^2 + b^2 - 2br\cos\theta}{Q^2}$$

Equate the terms independent of cosθ and coefficient of cosθ on both sides

$$a = \frac{r^2}{b}$$

$$\frac{Q'}{Q} = \frac{r}{b}$$

Image charge method : off centred point charge inside grounded sphere



Hollow sphere of radius r is kept at V=0
Inside the sphere there is a charge +Q placed at a distance a from the center.

What is the potential inside the sphere?

Notice the "conjugate“ nature of this problem with the last one.

This is a characteristic of "image charge problems".

How would you adapt the image charge method for a case where the spherical surface is at a potential V≠ 0 ?

Solving Laplace's equation

$$1\text{D}:(\textit{trivial !})$$

$$\frac{d^2V}{dx^2} = 0 \;\Rightarrow\; V = Ax + B$$

Cannot be anything more complicated.

$$2\text{D}:(\textit{cartesian})$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

In simplest (very few!) cases separation of variable will work.

$$\textit{If } V(x,y) = X(x)Y(y) \;\; \textit{then}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = k^2 \;\; (\textit{const})$$

$$\textit{Solution} = \text{sinusoidal type} \times \text{exponential decay or growth}$$

$$V(x,y) = \sum_{\textit{allowed } k} (A_k \cos kx + B_k \sin kx)(C_k e^{ky} + D_k e^{-ky})$$

A,B,C,D are chosen to match the given boundary conditions.

Role of x,y can be Interchanged.
X may be exponential &
Y may be sinusoidal.

Solving Laplace's equation (2D : Using complex numbers : corner)

Basic idea: Take any analytic complex function (eg. z^2, sin z, e^{-z})
F(z) = u(x,y) + i v(x,y)
Both u(x,y) and v(x,y) satisfy 2D Laplace equation

By intuition/guess/imagination make u(x,y) or v(x,y) satisfy the boundary conditions only. Uniqueness theorem guarantees that the guess is THE solution.

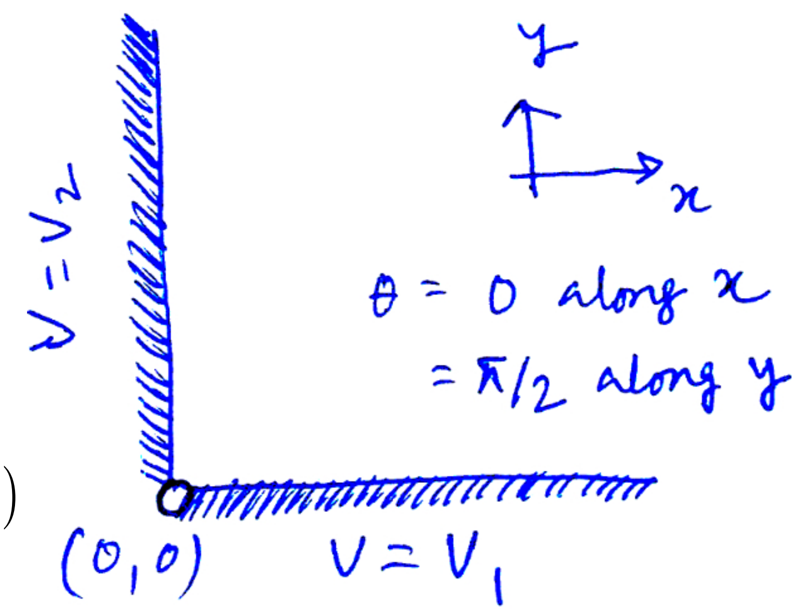
In reality, very few problems can be solved by separation of variables.
Quite a few can be done by the complex number method – but in 2D only.

Particularly useful for solutions in near corners, slits, edges, quadrupoles.

$$F(z) = i \ln z^2$$
$$u(x,0) = 0$$
$$u(0,y) = \pi$$

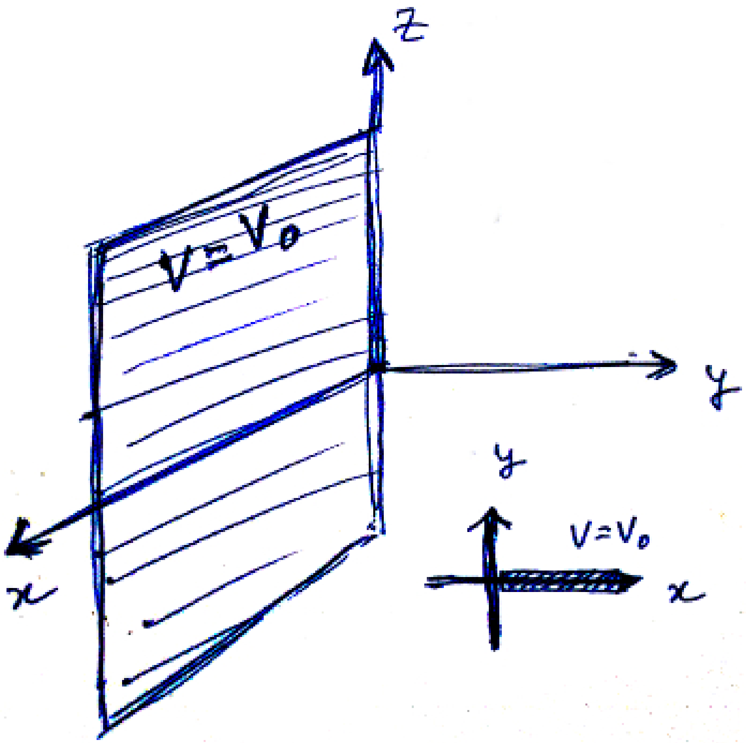
Use this fact and scale the function as

$$F(z) = -(V_2 - V_1)\, i \,\ln z^2 + V_1$$
$$u(x,y) = 2\frac{V_2 - V_1}{\pi} \arctan\left(\frac{y}{x}\right) + V_1 \quad (solved\,!)$$



Notice that separation of variable doesn't work here.
How will you modify the solution if the two plates are inclined at an angle α ?

Solving Laplace's equation (2D : Using complex numbers : edge)



A semi-infinite plate occupies the region x>0 in the xy plane.

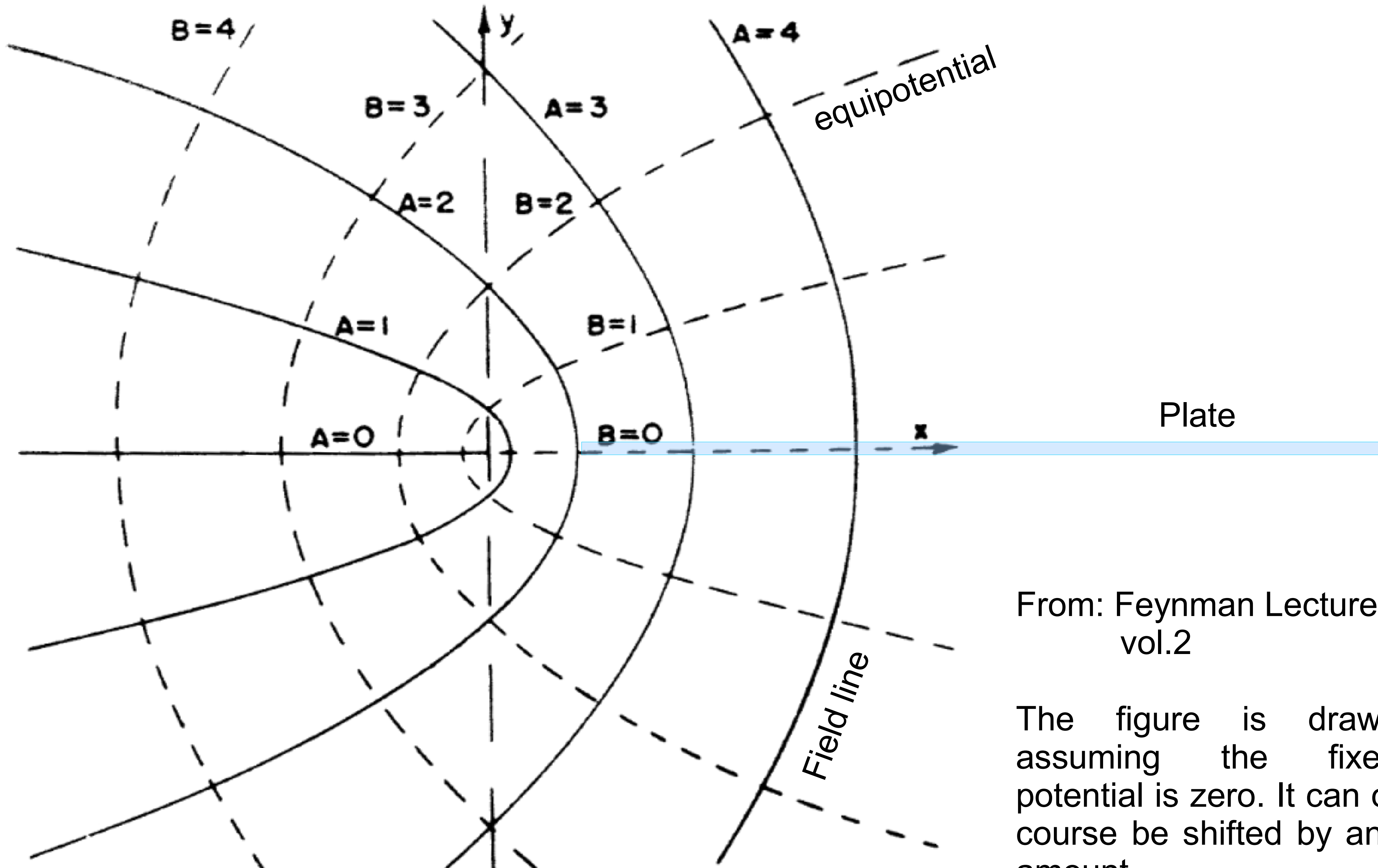
The plate is kept at V=const.

The problem shows how to handle the field near the edge of a flat thin plate.

$$F(z) = z^{1/2}$$
$$= \rho^{1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$
$$= \left[\frac{(x^2+y^2)^{1/2}+x}{2}\right]^{1/2} + i\left[\frac{(x^2+y^2)^{1/2}-x}{2}\right]^{1/2}$$

$$Notice \quad v(x,0) = 0 \quad if \quad x>0$$
$$v(x,y)+V_0 \quad satisfies \quad required \quad boundary \quad condition$$

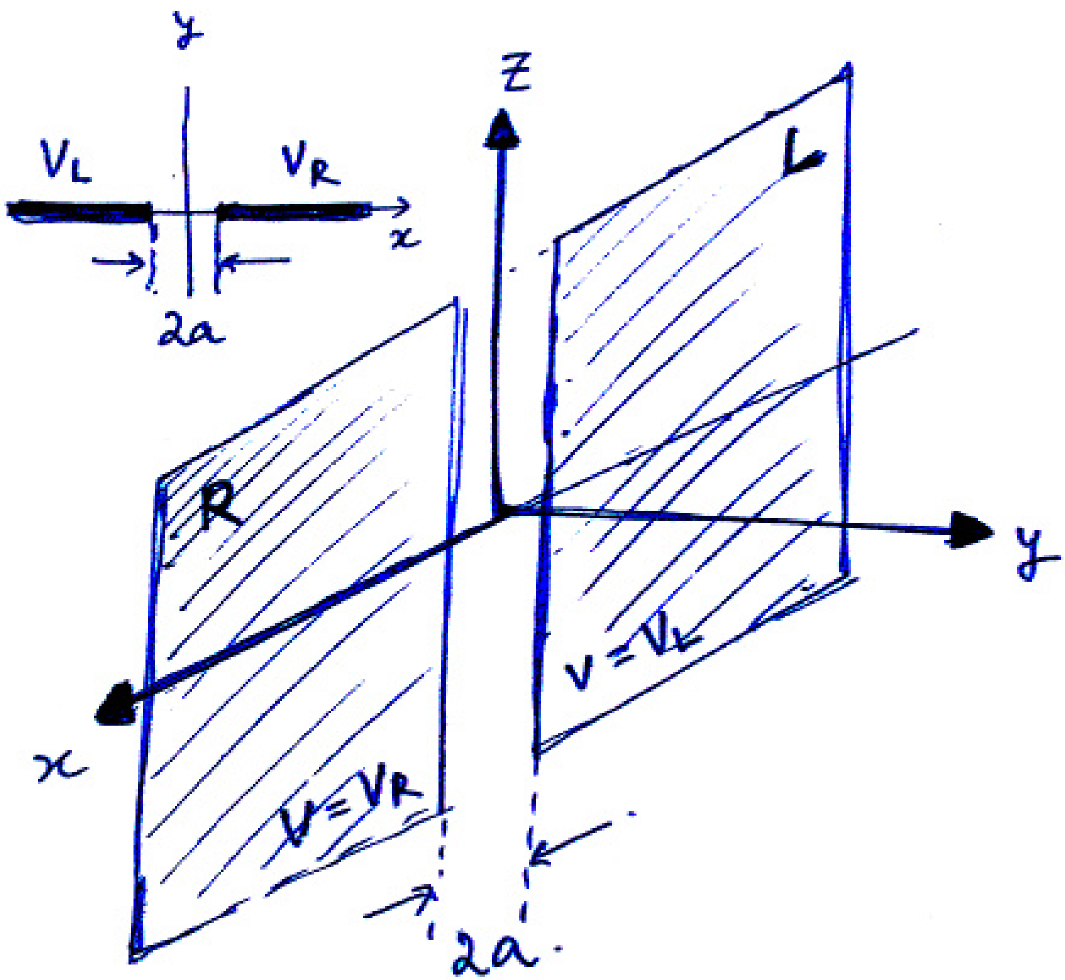
Solving Laplace's equation (2D : Using complex numbers : edge)



From: Feynman Lectures vol.2

The figure is drawn assuming the fixed potential is zero. It can of course be shifted by any amount

Solving Laplace's equation (2D : Using complex numbers : slit)



Two semi-infinite platea occupies the region [-a < x < a] in the xz plane.

The plates are kept at VL and VR

How to handle the field in a slit between equipotential plates?

The solution of this problem requires a transformation of the complex variables called "conformal transform".
See: Pipes & Harvill

You cannot superpose two plates at fixed potentials to get a slit. Why?

$$v(x,y)=V_L+\frac{V_R-V_L}{\pi}\left[\arcsin\frac{1}{2}\left(\sqrt{(x/a+1)^2+(y/a)^2}-\sqrt{(x/a-1)^2+(y/a)^2}\right)+\frac{\pi}{2}\right]$$

Solving Laplace's equation (2D : Plane polar)

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

This gives:

$$r^2\frac{d^2 R}{d r^2} + r\frac{dR}{dr} - m^2 R = 0$$

trial solution $R = A r^n$ *gives:* $n = \pm m$, *so*

$$V(r,\theta) = \sum_m \left(A_m r^m + \frac{B_m}{r^m}\right) e^{i m\theta}$$

Try: $V = R(r) e^{i m \theta}$

Why not $e^{m\theta}$?

Why should m be an integer?

What type of problems can we solve with this form?

Values given on a circle.
Solution inside should not have 1/r type solution
Solution outside (till infinity) should not have r type solution.\
Use Fourier analysis to find the coeffcients.

Solving Laplace's equation (3D : Spherical polar)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

With no ϕ dependence we try: $V(r,\theta) = R(r)\,P(\theta)$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = -\frac{1}{P}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = l(l+1)$$

The radial solution

$$r^2\frac{d^2 R}{dr^2} + 2\mathrm{r}\frac{dR}{dr} - l(l+1)R = 0$$

$$try \quad R = Ar^n$$

$$R = Ar^l + \frac{B}{r^{l+1}}$$

Notice the utility of wiritng the seperation constant in the l(l+1) form

Solving Laplace's equation (3D : Spherical polar)

The angular part:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)+l(l+1)P=0$$

Polynomial solutions worked in the examples before this, but would not work in this case. Why?

substitute $x=\cos\theta$

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+l(l+1)P=0$$

If we had kept the $e^{im\phi}$ dependence:

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\left(l(l+1)-\frac{m^2}{1-x^2}\right)P=0$$

.... In atomic wavefunctions it is common

try the series: $P=\sum_0^\infty a_n x^n$: this gives

$$(1-x^2)\sum n(n-1)a_n x^{n-2}-2x\sum n a_n x^{n-1}+l(l+1)\sum a_n x^n=0$$

$$2a_2+l(l+1)a_0=0$$

$$3.2.a_3-2a_1+l(l+1)a_1=0$$

$$a_{n+2}=-\frac{(l-n)(l+n+1)}{(n+2)(n+1)}a_n$$

a0 and a1 can be arbitrarily chosen

If l is an integer, then the series will terminate at n=l

Odd and even powers do not mix in this recurrence relation

Solving Laplace's equation (3D : Spherical polar)

$$P(x) = a_0 \sum (\text{even powers of } x) + a_1 \sum (\text{odd powers of } x)$$

So construct each polynomial using the recurrence relation

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Legendre Polynomials:

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1} \delta_{nm}$$

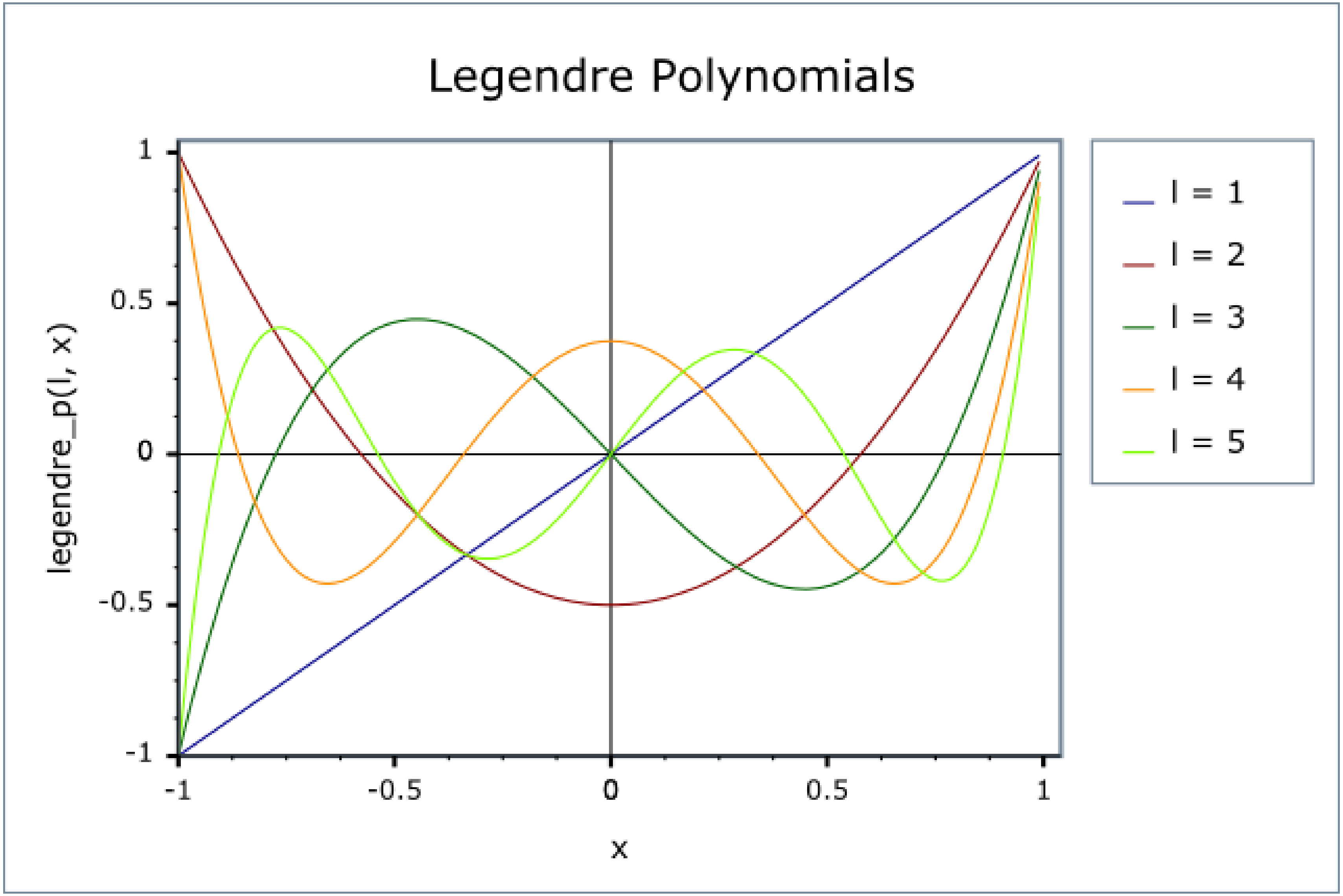
Use orthogonality
to find expansion co-effs ...

$$V(r, \theta) = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Solutions of a general class of diffn equations have this orthogonality property – called "Sturm-Liuville“ diffn eqn

Values given on a sphere.
Solution inside should not have 1/r type solution
Solution outside (till infinity) should not have r type solution.
Use expansion in Legendre Polynomials to find the coeffcients.
See the worked examples of Griffiths...section 3.3

Solving Laplace's equation (3D : Spherical polar)



To generate the successive $P_l(x)$ use the Rodrigue's formula :

$$P_l(x) = \frac{1}{2^l\, l\,!}\frac{d^l}{dx^l}\left(x^2-1\right)^l$$

Work and Energy in electrostatics

Conservative field : Total energy of a particle is conserved.

✔KE+PE is conserved. Or equivalently

✔Work done in moving a particle very slowly from one point to another is path independent.

✔A potential energy function exists

✔The force is derivable from a scalar potential

✔Curl of the Force field is zero.

Gravitational potential:

Apple falling from a tree
Earth going round the sun...
Trajectory of a particle...

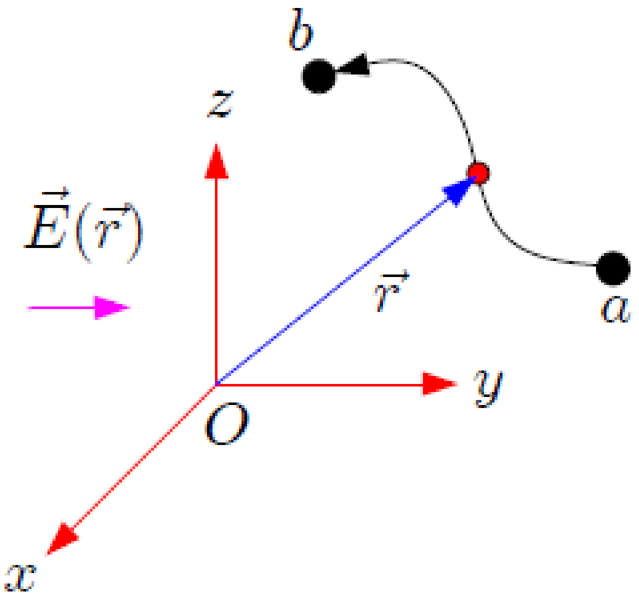
Why do we need to say more?
The answer to this is not within "electrostatics"....the need really comes
When we deal with E,B and moving charges.

Work and Energy in electrostatics

Work done on the charge

$$W = -q\int_a^b \boldsymbol{E}\,.\,\boldsymbol{dr} \quad [\text{always true}]$$

$$= q\left(V_b - V_a\right) \quad [\text{only if } \nabla \times \boldsymbol{E} = 0]$$



SI unit is Joule.
Useful unit is electron-Volt
Work needed to move one electron through 1 volt
$q=-1.6 \times 10^{-19}$C

How do we build up a configuration of charges?

Bring the first charge : No work done

Bring the second charge from infinity to desried position : calculate work done

Bring next one. Calculate the work done due to the presence of the previous TWO

Work and Energy in electrostatics

$$W = \frac{1}{4\pi\epsilon_0}\sum_{j<i}\frac{q_i q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} = \frac{1}{2}\sum_i q_i \boxed{\frac{1}{4\pi\epsilon_0}\sum_j \frac{q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}}$$

Potential at location of charge i

Now suppose this is a continous distribution : it means we are saying the following

$$\sum_i q_i(.....) \longrightarrow \int_{all\,space} \rho(\boldsymbol{r})\, d\tau (.....)$$

When can this breakdown?

$$\frac{1}{2}\int \rho V\, d\tau = \frac{\epsilon_0}{2}\int (\nabla.\boldsymbol{E}) V\, d\tau$$

$$= \frac{\epsilon_0}{2}\int [\nabla.(\boldsymbol{E}V) - \boldsymbol{E}(-\nabla V)]\, d\tau$$

$$= \frac{\epsilon_0}{2}\int [\boldsymbol{E}.\boldsymbol{E}]\, d\tau$$

Convert to surface integral
Take surface at infinity
Should go to zero

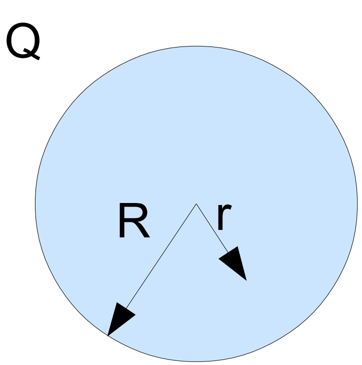
Energy of a point charge diverges!

$$\frac{\epsilon_0}{2}\int_0^{\infty}\left(\frac{q}{4\pi\epsilon_0 r^2}\right)d\tau \qquad \text{does not converge}$$

The closest we can try :

Assume that a point charge is a uniform sphere of some radius R.

The integral for field energy will then converge.



$$\rho=\frac{Q}{\frac{4}{3}\pi r^3}$$

$$E_{in} = \frac{Q}{4\pi\epsilon_0 R^2}\frac{r}{R} \qquad (r<R)$$

$$E_{out} = \frac{Q}{4\pi\epsilon_0 r^2} \qquad (r>R)$$

$$\left(\frac{\epsilon_0}{2}\right)\left[\int_0^R E_{in}^2\, d\tau+\int_R^{\infty} E_{out}^2\, d\tau\right] \qquad \text{will converge}$$

Within classical electromagnetism it is not possible to resolve this problem.

We have to accept that the concept of a point charge has some limitations

Dielectric materials

Dielectric materials:

Field of a polarised object at a large distance
Multipole expansion of scalar potential
Polar and cartesian expressions for dipole, quadrupole etc
Atomic and molecular origin of the dipole moment
Equivalent charge distribution
Force and torque on a dipole
Definition of the E D P vectors and boundary conditions
Interface of two dielectrics, sphere in an uniform field
Energy contained in Electric fields with dielectrics present

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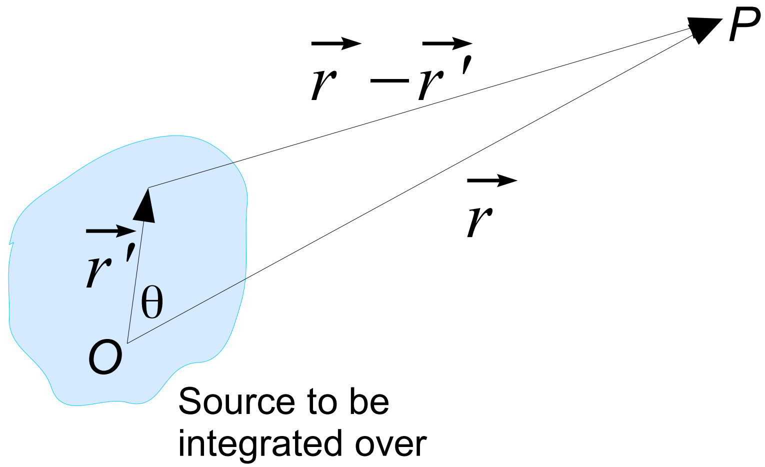
How does a charge distribution look from far away?

Quantitatively this means : With what power law does it fall off ….inverse square, cube, fourth ?

Often the charge is limited to a small area.



Answer :

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

In many cases $r \gg r'$

So expand in a power series in $\dfrac{r'}{r}$

From the figure :

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \left[r^2 + r'^2 - 2rr'\cos\theta\right]^{-\frac{1}{2}}$$

$$= \frac{1}{r}\left[1 - \left\{2\frac{r'}{r}\cos\theta - \left(\frac{r'}{r}\right)^2\right\}\right]^{-\frac{1}{2}}$$

$$= \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta)$$

Legendre polynomials again!

Binomial expansion .....

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 \ldots$$

Multipole expansion of the electrostatic potential

$$V(P) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int d^3\vec{r}\,'\left[\rho(\vec{r}\,')\,r'^{\,l}\,P_l(\cos\theta)\right]$$

monopole

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int d^3\vec{r}\,'\rho(\vec{r}\,') \;+\right.$$

dipole

$$\frac{1}{r^2}\int d^3\vec{r}\,'\,r'\cos\theta\,\rho(\vec{r}\,') \;+$$

quadrupole

$$\left.\frac{1}{r^3}\int d^3\vec{r}\,'(r')^2\frac{1}{2}(3\cos^2\theta-1)\rho(\vec{r}\,') \;+\;\ldots\right]$$

If the total charge is zero: Dipole term dominates.

If that is also zero Quadrupole dominates

suppose

$$\rho(\vec{r}\,') = q\,\delta(\vec{r}\,'-\vec{a}) - q\,\delta(\vec{r}\,'+\vec{a})$$

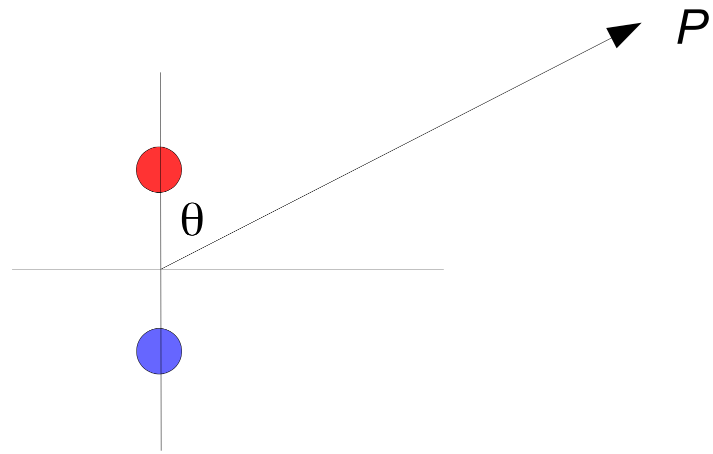
how will the dipole integral look?

can write this as

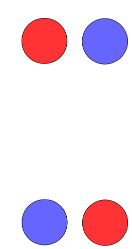
$$V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{\hat{r}}}{r^2}$$

with $\vec{p} = \sum q_i \vec{r}_i\,'$

and some other equivalent forms...



Stick two monopoles to get a dipole.



Stick two dipoles to get a quadrupole.

Choice of the co-ordinate system and origin in multipole expansion

We could have done the expansion in a more cartesian way...

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = [r^2+r'^2-2\vec{r}.\vec{r}\,']^{-\frac{1}{2}}$$

This would have given succeessive terms like....

$$V_{mono} = \frac{1}{4\pi\epsilon_0}\frac{Q_{total}}{r}$$

$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\sum \hat{r}_i p_i}{r^2}$$

$$V_{quad} = \frac{1}{4\pi\epsilon_0}\frac{1}{2}\frac{\sum_{ij} \hat{r}_i \hat{r}_j Q_{ij}}{r^3}$$

$$p_i = \int d^3\vec{r}\,' r_i' \rho(r')$$

$$Q_{ij} = \int d^3\vec{r}\,'\left(r_i' r_j' - r'^2\delta_{ij}\right)\rho(r')$$

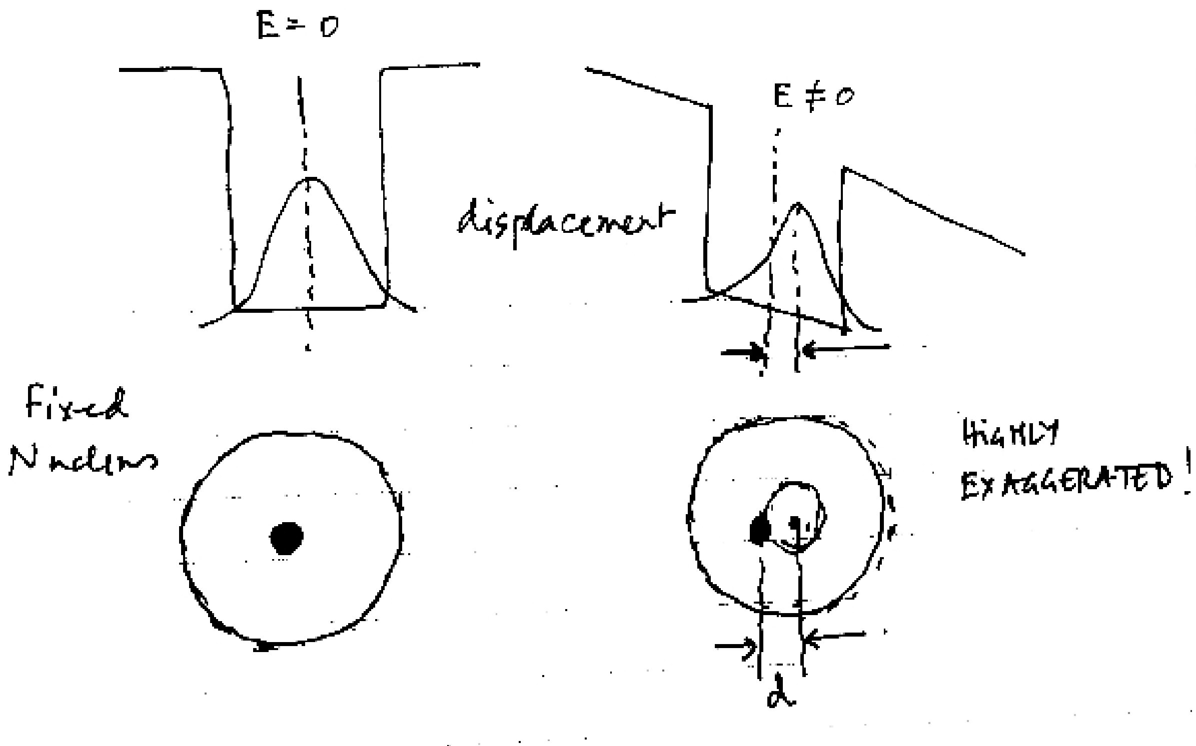
Dipole moment is a vector
Quadrupole moment is a tensor

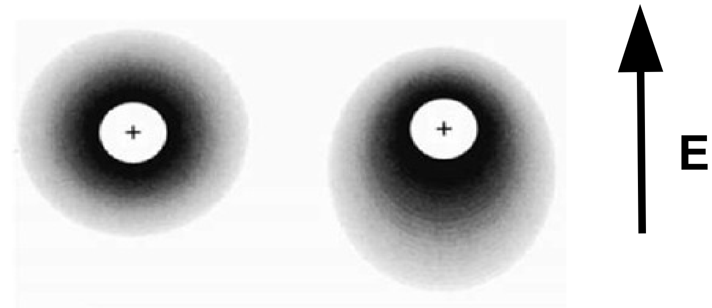
The lowest non-vanishing moment is independent of the choice of the origin.
The higher moments are NOT necessarily so.

So if the total charge (monopole) is zero then dipole term is origin-independent.
If the dipole also vanishes then quadrupole is origin independent. (Prove it!)

Dipole is more common in electronic charge distributions.
Nucleii often have quadrupole moments.
Earth's gravitational potential has a significant quadrupole component.

Response of atoms and molecules to an electric field





Q is the total charge
A is the atomic radius
p = Qd (induced moment)

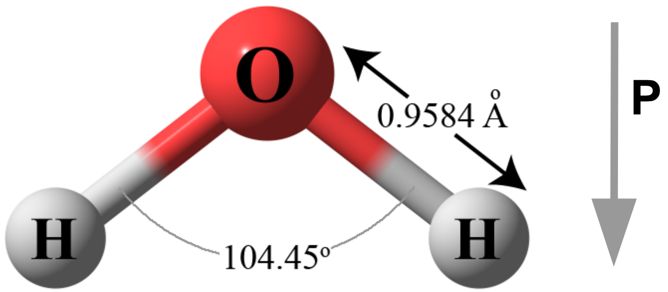
Electron cloud is an uniformly charged sphere..(assume)
Force on the nucleus due to displaced electron cloud = External force on nucleus

$$E = \frac{1}{4\pi\epsilon_0}\frac{Qd}{a^3} \qquad hence \qquad \vec{p} = 4\pi\epsilon_0 a^3 \ \vec{E}$$

$$\sim \ 4\pi\epsilon_0 \times 10^{-30} \quad \text{in SI}$$

Atomic polarizability
Small for inert gases
Large for atoms with partially filled outer shell
Estimated values and observed value agree (order of magnitude)

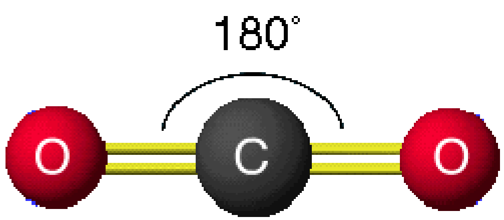
Atoms and molecules in an electric field: frozen moment of molecules



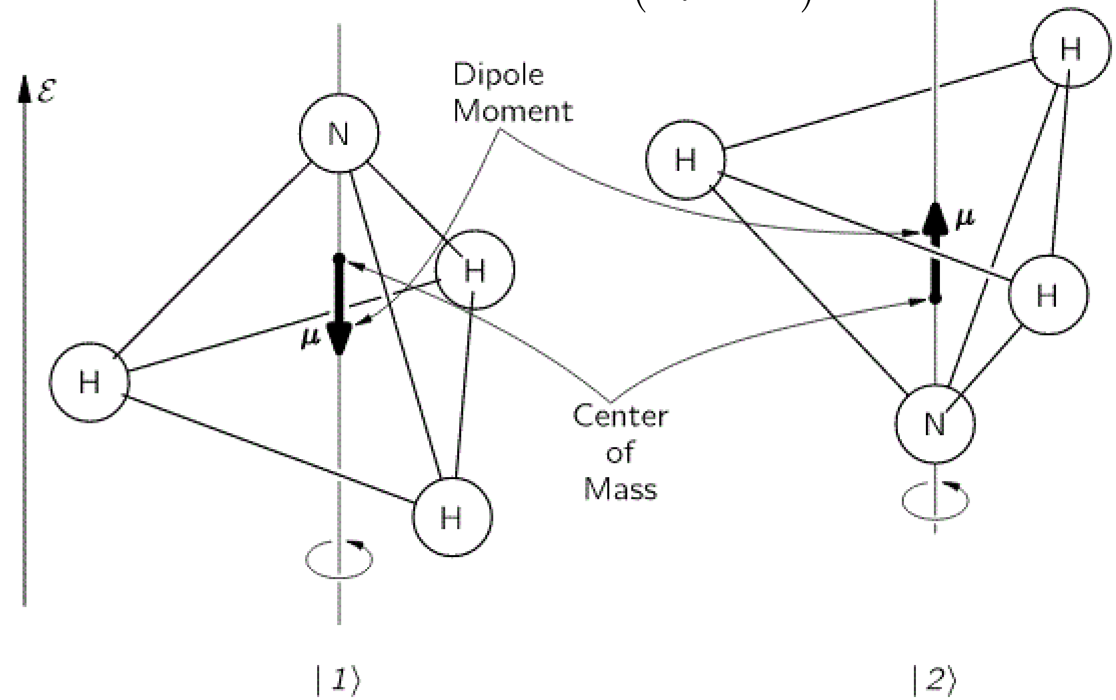
Electron distribution in the bonds can give rise to built in dipole moment

$$4.8\times10^{-30}\,Coul.mt$$
$$(1.42\,D)$$

$$6.2\times10^{-30}\,Coul.mt$$
$$(1.85\,D)$$



No dipole moment



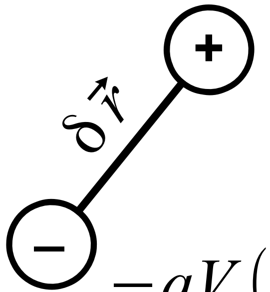
SI unit = Coul-mt.

1 Debye unit (historical but useful)
Dipole moment of 10^{-10} esu of charge separated by 1 angstrom
Useful for molecular scale since
Electron charge is 4.8 $\times 10^{-10}$ esu

Induced dipole moment and electric field are not necessarily in the same direction for a molecule. Since the bonds do not shift uniformly in all directions...."easy" and "hard" directions...
P and E are related by a matrix/tensor

Force and torque on a dipole

Potential Energy
and force



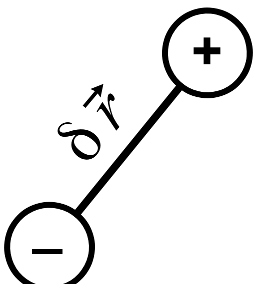
$$qV(\vec{r}+\delta\vec{r})=q\left(V+\delta\vec{r}\,.\,\vec{\nabla}V\right)$$

$$-qV(\vec{r})$$

Add the two contributions

$$\begin{aligned}\vec{p} &= q\,\delta\vec{r}\\ U_{dip} &= -\vec{p}\,.\,\vec{E}\\ \vec{F}_{dip} &= \left(\vec{p}\,.\,\vec{\nabla}\right)\vec{E}\end{aligned}$$

Torque



$$(\vec{r}+\delta\vec{r})\times[q\,\vec{E}(\vec{r}+\delta\vec{r})]$$

$$\vec{r}\times[-q\,\vec{E}(\vec{r})]$$

$$\vec{\tau}_{dip} = \vec{p}\times\vec{E}$$

Although the E field is different at two sites, the difference in the final expression would be second order.....

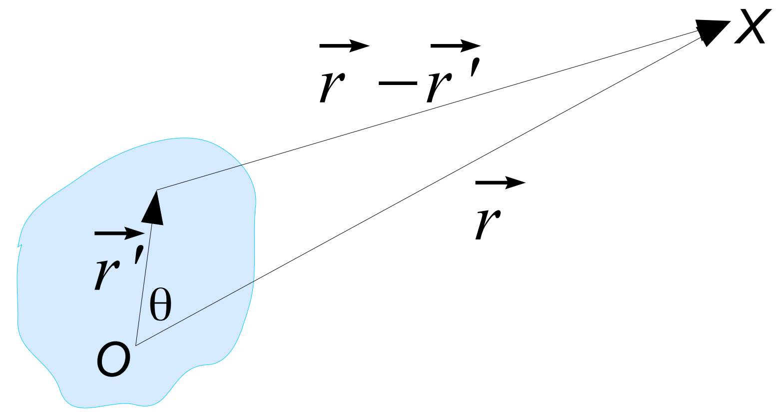
Now we can calculate the interaction force between two dipoles....easily!
If we have two dipoles...the E field of the first will act on the second and vice versa,

Potential of an extended distribution of dipoles

$$V(X) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\vec{P}.\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$$\nabla_{r'}\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

Prove this by writing out in (x-x')…..



Hence

$$V(X) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\left[\nabla\frac{\vec{P}}{|\vec{r}-\vec{r}\,'|} - \frac{1}{|\vec{r}-\vec{r}\,'|}\nabla.\vec{P}\right]$$

dipole distribution to be integrated over

$$= \frac{1}{4\pi\epsilon_0}\left[\int\frac{d\vec{S}\,'.\vec{P}}{|\vec{r}-\vec{r}\,'|} - \int d^3\vec{r}\,'\frac{\nabla.\vec{P}}{|\vec{r}-\vec{r}\,'|}\right]$$

Here integration and differentiation are w.r.t. primed co-ordinates

Surface charge $\sigma = \vec{P}.\hat{n}$

Volume charge $\rho = -\nabla.\vec{P}$

Linear Dielectrics : **E P D** vectors

Linear dielectric means : Induced dipole moment (**P**) is proportional to the electric field. Hence:

$$\nabla . \vec{E} = \frac{\rho_{TOTAL}}{\epsilon_0}$$

$$\nabla . \epsilon_0 \vec{E} = \rho_{free} + \rho_{pol} \qquad \left(since \quad \rho_{pol} = -\nabla . \vec{P} \right)$$

$$\nabla . \left[\epsilon_0 \vec{E} + \vec{P} \right] = \rho_{free}$$

$$Use\ the\ proprotionality\ of\ \vec{P}\ with\ \vec{E}: \quad \vec{P} = \epsilon_0 \chi \vec{E}$$

$$\epsilon_0 (1 + \chi) \vec{E} = \epsilon \vec{E} = \vec{D}$$

susceptibility

Historically called electric displacement vector: Microscopic mechanism was not known then.

$$\nabla . \vec{D} = \rho_{free}$$

$$\nabla . \vec{E} = \frac{\rho_{free}}{\epsilon}$$

$$\nabla \times \vec{E} = 0$$

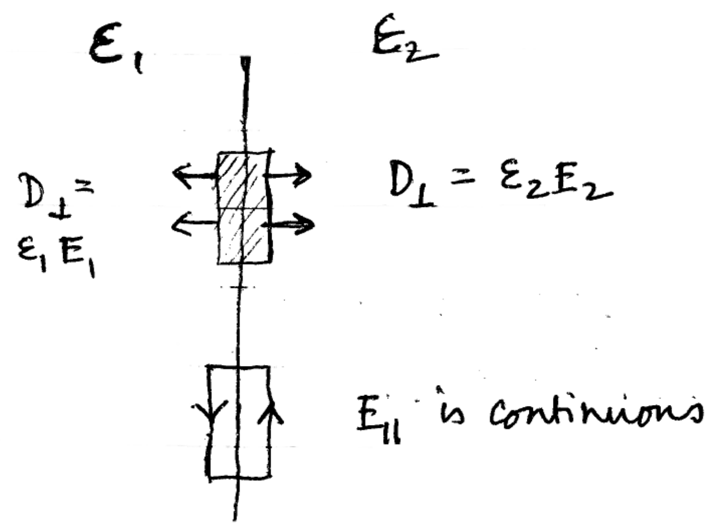
Quantities like D, ε can only be defined in an average sense.

!! One cannot talk about D or ε inside an atom!!

These only make sense if averaged over a few (~10 -100) lattice units.

Since curl **E** =0, a scalar potential is still possible.
But the "source" of this potential is reduced by a factor.
Hence the scalar potential V is also reduced by that factor.

Linear Dielectrics : **E P D** vectors : Boundary conditions and related problems



No FREE CHARGE

$\nabla . D = \rho_{free} = 0$

$$\begin{aligned}
\rho_{free} &= 0 && (at \quad the \quad interface) \\
\epsilon_1 E_1^{\perp} &= \epsilon_2 E_2^{\perp} && (since\, \nabla . \vec{D} = 0) \\
E_1^{\parallel} &= E_2^{\parallel} && (since\, \nabla \times E = 0)
\end{aligned}$$

Point charge q is placed at (0,0,d) as shown near an interface of two dielectrics.
Q: What is the potential everywhere?

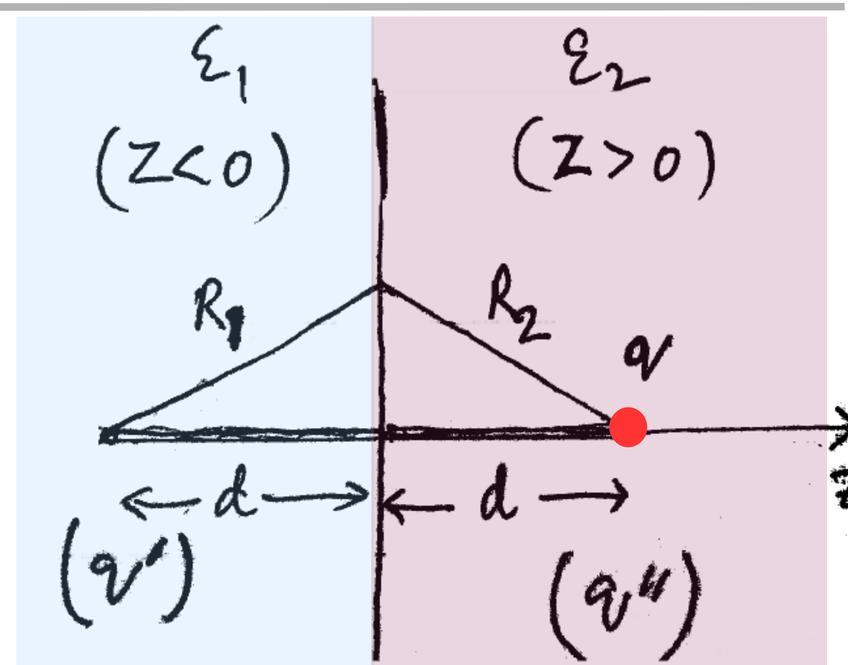
For z >0 : (region 2)
Image charge q' at (0,0,-d)

For z<0 : (region 1)
Charge q'' at (0,0,d)

Write the potential, then the electric field. Two independent equations by matching the normal and tangential components at the boundary.

$$q' = -\left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right) q$$

$$q'' = \left(\frac{2\epsilon_1}{\epsilon_1 + \epsilon_2}\right) q$$

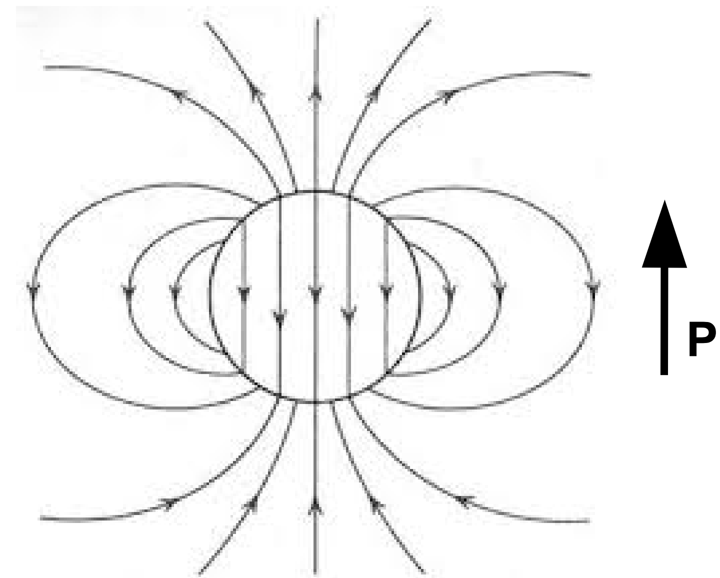


$\boxed{\epsilon_1 \nabla . \vec{E}_1 = 0}$ $\boxed{\epsilon_2 \nabla . \vec{E}_2 = \rho}$

Linear Dielectrics : A uniformly polarised sphere

Uniformly polarised sphere : (no external field)

Note the lines of force (Electric field):
Points in the opposite direction inside the sphere.



$$V(r,\theta)=\begin{cases}\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) & (0<r\leqslant R)\\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) & (r\geqslant R)\end{cases}$$

$$\begin{aligned} Surface\,charge &: \sigma_b = \vec{P}.\hat{n} = P\cos\theta \\ Volume\,charge &: \rho_b = -\nabla.\vec{P} = 0 \end{aligned}$$

Use boundary conditions at r=R and
V should be well behaved at small and large r...

V(r=R) should match
E should have a discontinuity due to surface charge
Equate the coefficient of each Legendre polynomial

$$V(r,\theta)=\begin{cases}\frac{P}{3\epsilon_0} r\cos\theta & (0<r\leqslant R)\\ \frac{P}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta & (r\geqslant R)\end{cases}$$

Looks like the field of a single (pure) dipole at r=0

The field is CONSTANT inside

Linear Dielectrics : A dielectric sphere in an uniform field

Uniform field means far from the sphere $E = E_0$ set externally

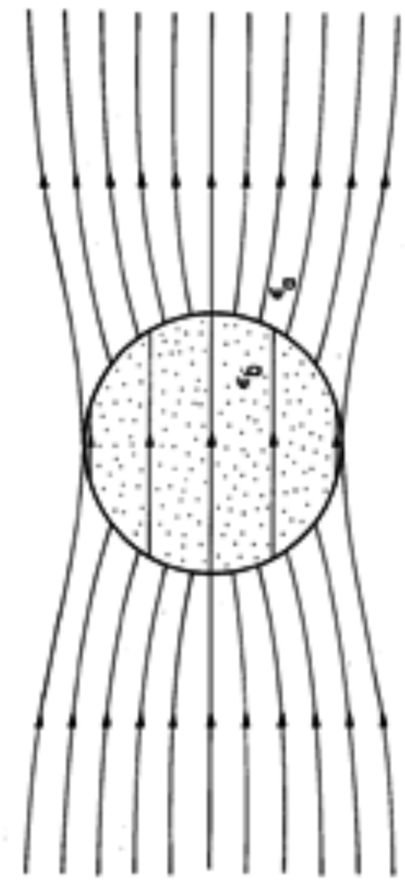
$$V(r,\theta)=\begin{cases}\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) & (0<r\leqslant R)\\ \sum_{l=0}^{\infty}\left[B_l r^l+\frac{C_l}{r^{l+1}}\right]P_l(\cos\theta) & (r\geqslant R)\end{cases}$$

$$-\epsilon\frac{\partial V_{\text{in}}}{\partial r}\bigg|_{r=R} = -\epsilon_0\frac{\partial V_{\text{out}}}{\partial r}\bigg|_{r=R}$$

Normal component of D is continuous

$$-\frac{1}{R}\frac{\partial V_{\text{in}}}{\partial\theta} = -\frac{1}{R}\frac{\partial V_{\text{out}}}{\partial\theta}$$

Tangential component of E is continuous



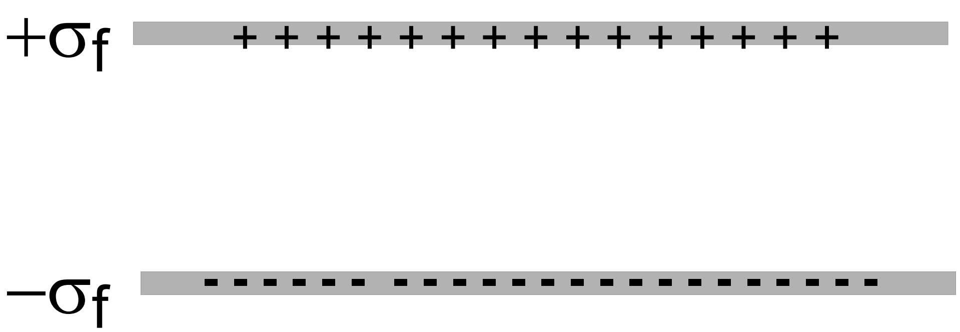
$$V_{\text{in}} = \left(\frac{3}{2+\epsilon/\epsilon_0}\right)E_0 r\cos\theta$$

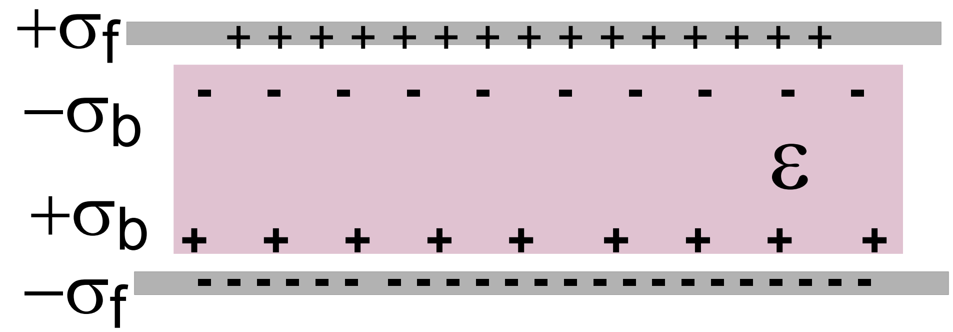
$$V_{\text{out}} = -E_0 r\cos\theta+\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)E_0\frac{R^3}{r^2}\cos\theta$$

What would $\epsilon\rightarrow\infty$ physically mean?
If a spherical cavity is dug out in a large slab?

Notice how the orthogonality of Legendre polynomials is crucial to solving these problems

Linear Dielectrics : A capacitor with a dielectric slab





$$\frac{V}{d} = E = \frac{\sigma_f}{\epsilon_0}$$

$$\frac{V}{d} = E = \frac{\sigma_f - \sigma_b}{\epsilon_0}$$

$$\sigma_b = \vec{P}\,.\,\hat{n} = \epsilon_0 \chi\, E$$

$$so \quad E = \frac{\sigma_b}{\epsilon_0 \chi}$$

$$\sigma_f - \sigma_b = \frac{\sigma_f}{1+\chi}$$

$$\frac{V}{d} = E = \frac{\sigma_f}{\epsilon_0 (1+\chi)}$$

For the same charge on the metal plate, the voltage developed is now smaller.

C=Q/V thus increases by a factor of 1 + χ

The energy stored in the field also increases by the same factor if the same voltage and hence the same E=V/d is established in the capacitor.

$$\boxed{W = \frac{\epsilon_0}{2}\int d^3\vec{r}\;\; \vec{E}\,.\,\vec{E} \quad \rightarrow \quad W = \frac{1}{2}\int d^3\vec{r}\,\vec{E}\,.\,\vec{D}}$$

Magnetostatics

Magnetostatics: Field due to steady currents

Magnetic fields are created by:

1. Charges in motion
2. Intrinsic spin (dipole) moments of some elementary particles.

The field created by a single moving charge is not very easy to write down!
Historically steady currents were understood first.

Maxwell's equation tell us the field created by currents (moving charges).
The response of a particle to the fields (E and B) is an independent input.

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

<u>Lorentz Force Law is Not derivable from Maxwell's equation</u>

$$\nabla . \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

= 0 for now

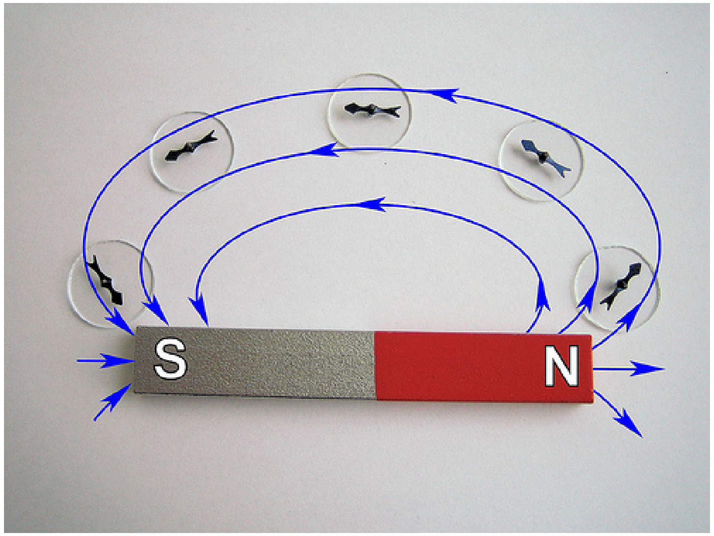
Historically these equations were written later.

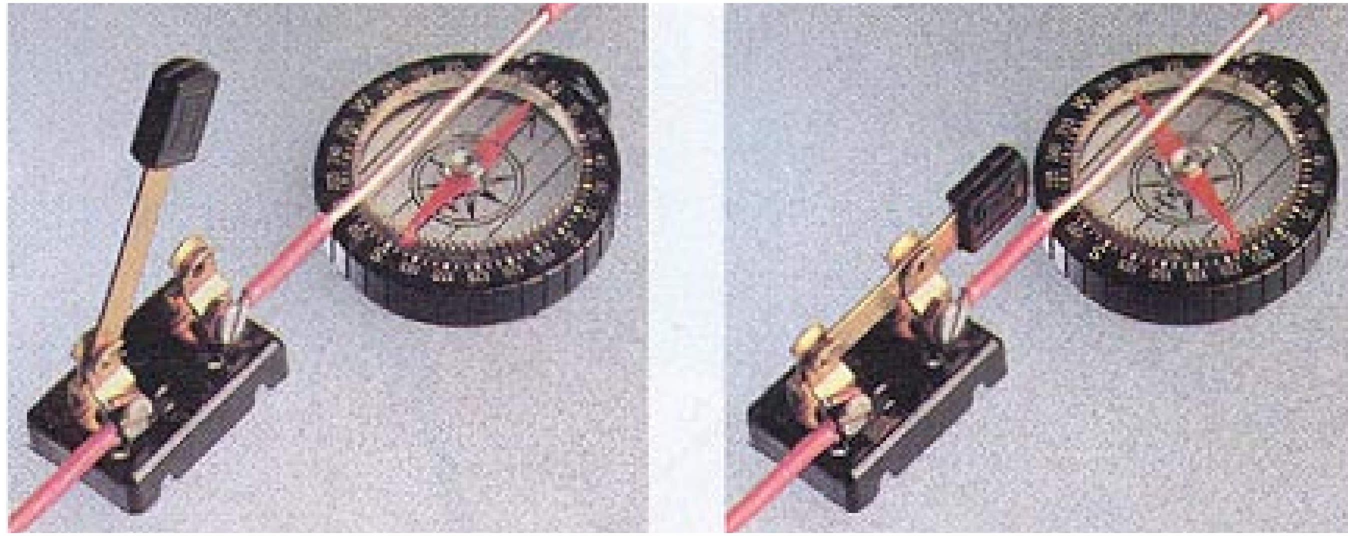
Recall that curl and div specifies a vector field fully if suitable boundary conditions are also given.

Zero divergence means :
No analogue of charge as in electrostatics.

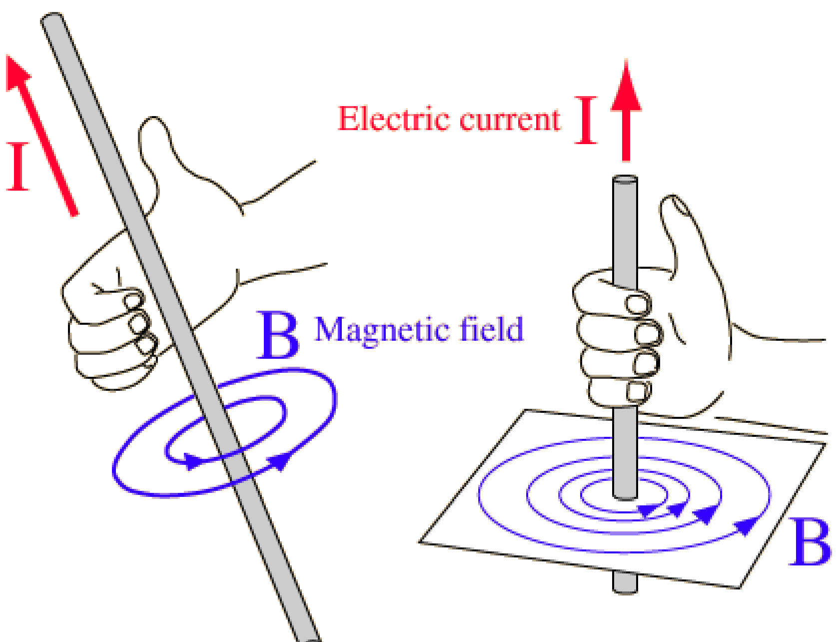
$$\mu_0 = 4\pi \times 10^{-7}\, Henry.m^{-1}$$

Histroical observations, Biot Savart Law





Magnets deflect a "compass" and so does a nearby current carrying wire. So a current carrying wire must be creating a magnetic field. (Oersted, Ampere)

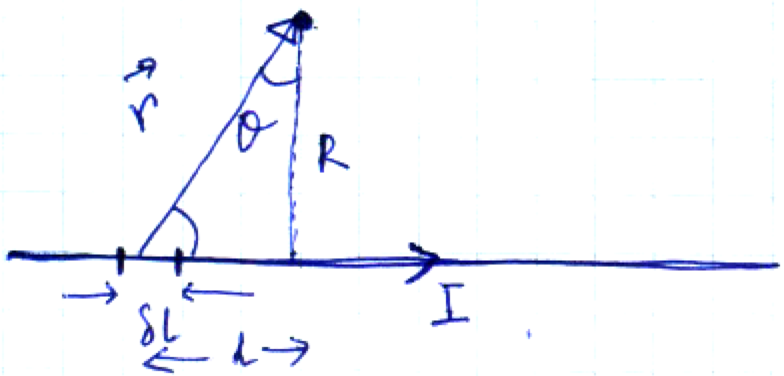


$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\vec{r}}{r^3}}$$

Then integrate over the wire to find the full field.

Common geometries.....
Straight lines, coils etc.

Field from a wire segment, loops, coils, toriods etc



$$R = r\cos\theta$$

$$l = R\tan\theta$$

$$\delta l = R\sec^2\theta\, \delta\theta$$

$$\delta B = \frac{\mu_0}{4\pi} I \frac{\vec{\delta l} \times \vec{r}}{r^3}$$

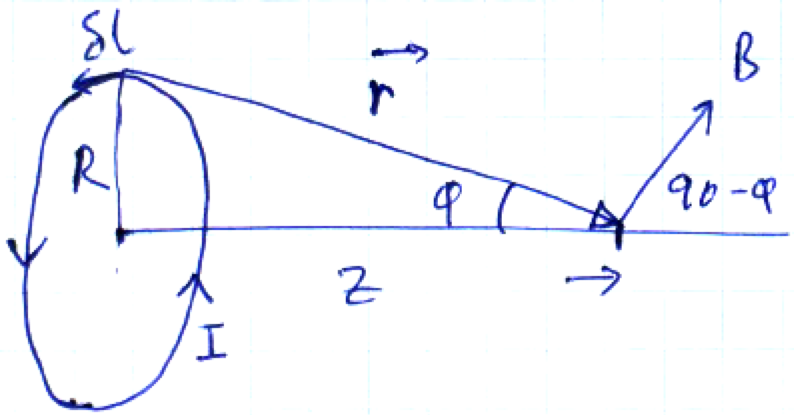
$$= \frac{\mu_0 I}{4\pi} \cdot \frac{R\sec^2\theta\,\delta\theta \,.\, \sin(\pi/2 - \theta) \,.\, R\sec\theta}{R^3\sec^3\theta}$$

$$B = \frac{\mu_0 I}{4\pi R} \int_{\theta_1}^{\theta_2} \cos\theta\, d\theta = \frac{\mu_0 I}{4\pi} (\sin\theta_2 - \sin\theta_1)$$

Infinite wire $\begin{cases} \theta_1 = -\pi/2 \\ \theta_2 = \pi/2 \end{cases}$

$$B = \frac{\mu_0 I}{2\pi R} \quad \text{(Can use symmetry arguments)}$$

Field from a wire segment, loops, coils, toriods etc



$$r^2 = R^2 + z^2$$

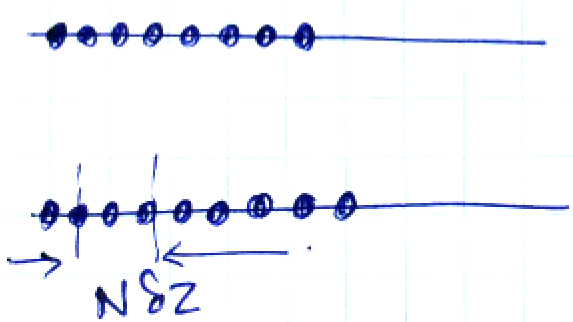
$$\sin\varphi = R/r$$

$$\delta l = R\,\delta\theta$$

$$\delta B = \frac{\mu_0}{4\pi} \cdot I(R\,\delta\theta) \cdot \frac{1}{R^2+z^2} \cdot \frac{R}{r}$$

$$B = \frac{\mu_0 I}{2} \cdot \frac{R^2}{(R^2+z^2)^{3/2}}$$

Integrate over z for a finite coil.

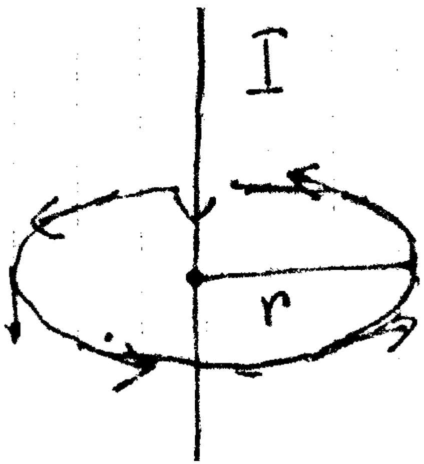


$$dB = \frac{\mu_0 I}{2} \frac{R^2\, N\,\delta z}{(R^2+z^2)^{3/2}}$$

Using the integral form in symmetrical cases: long wire, coil, full toroid

$$\oint \vec{B} . \vec{dl} = \mu_0 I_{enc} \quad \text{same as} \quad \nabla \times \vec{B} = \mu_0 \vec{J}$$

Infinitely long wire



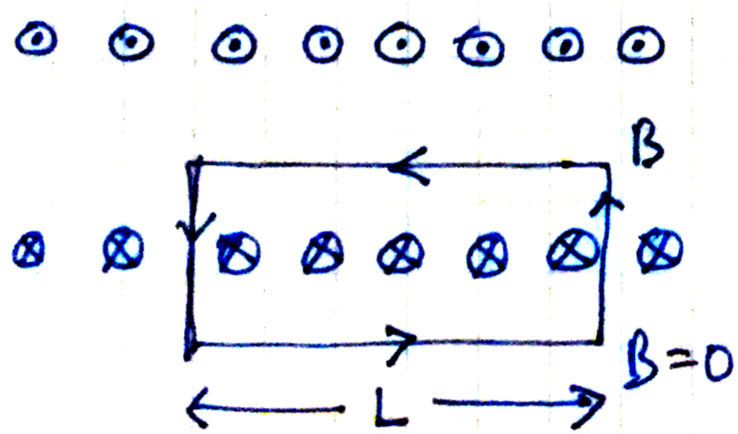
$$B_\varphi 2\pi r = \mu_0 I$$

$$B_\varphi = \frac{\mu_0 I}{2\pi r}$$

$$B_r = 0 \quad (\nabla . B = 0)$$

$$B_z = 0$$

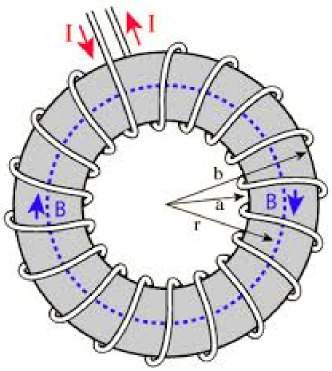
Infinitely long coil

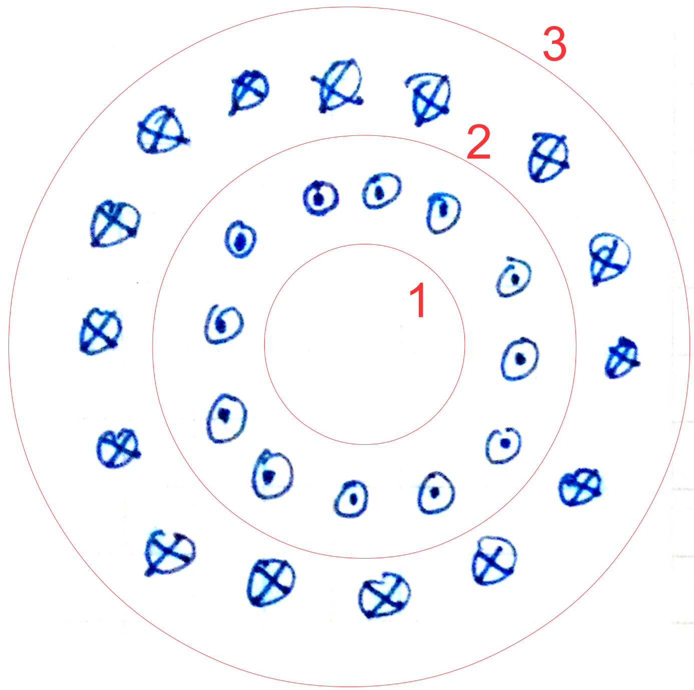


$$B L = \mu_0 N I$$

$$B = \mu_0 \frac{N}{L} I$$

Using the integral form in symmetrical cases: long wire, coil, full toroid





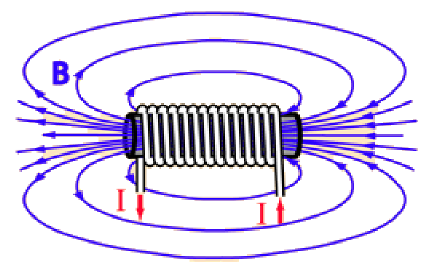
Path 1, 3 $\oint \vec{B}\cdot d\vec{l} = 0$

Path 2 $\oint \vec{B}\cdot d\vec{l} = \mu_0 N I.$

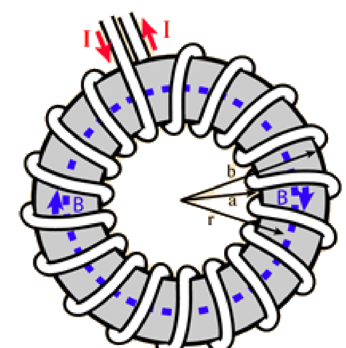
$$B_r = \frac{\mu_0 N I}{2\pi r}$$

Geometries where one can apply Ampere's Law quickly.

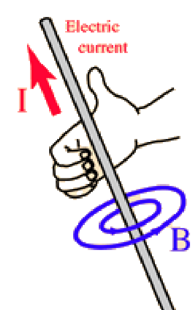
Similar to using Gauss's law in some symmetrical situations.



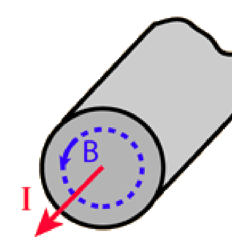
Magnetic field inside a long solenoic.



Magnetic field inside a toroidal coil.



Magnetic field from a long straight wire.



Magnetic field inside a conductor.

The magnetic vector potential : the formal solution

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$
$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J}$$
$$\nabla (\nabla . \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

The choice $\nabla . \vec{A} = 0$
is called a gauge choice

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

is like three Poisson's equation $\left(\nabla^2 V = -\frac{\rho}{\epsilon_0}\right)$ put together

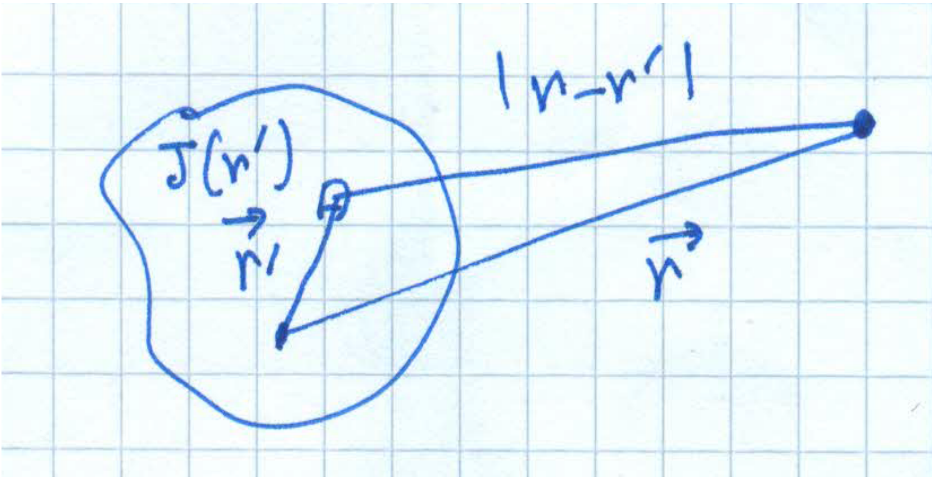
Since $\nabla . \vec{B} = 0$ we can write $\vec{B} = \nabla \times \vec{A}$
Does it make things simpler ?

There can be other possible choices.
For each gauge A and V will be different
But they will give the same **E** and **B**.

Call $\vec{A}$ the vector potential.
$\vec{A}$ has <u>no</u> simple interpretation like potential energy

The magnetic vector potential : the formal solution

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'}$$



$$\vec{B} = \nabla\times\vec{A}(\vec{r}) = \nabla\times\frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'$$

The curl is to be taken w.r.t. $\vec{r}$
The integration is w.r.t. $\vec{r}\,'$

We can interchange the order of integration and differentiation.

what is $\nabla\times\dfrac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$?

$$= \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{J_k(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$= \epsilon_{ijk} J_k(\vec{r}\,')\frac{\partial}{\partial x_j}\frac{1}{|\vec{r}-\vec{r}\,'|}$$

$$= \epsilon_{ikj} J_k(\vec{r}\,')\frac{x_j - x'_j}{|\vec{r}-\vec{r}\,'|^3}$$

$$= \left[\vec{J}(\vec{r}\,')\times\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right]_i$$

We recover Biot-Savart Law, which is an important consistency check!

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')\times\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\, d^3\vec{r}\,'}$$

The magnetic vector potential : the choice of div A and its consequences

Our choice of A cannot affect the final result for B.

But is does effect the solution for BOTH the scalar and the vector potential.

Notice that A,V suffer from "instantaneous change at a distance" problem.

We do not need to care as long as it is a static/steady state solution.

But what if charge and current densities (hence A and V) are both varying arbitrarily?

The choice of A and V : how much freedom is there?

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}\nabla\times\vec{A}$$

$$\nabla\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right) = 0$$

$$\vec{E}+\frac{\partial\vec{A}}{\partial t} = \nabla(some\ scalar)$$

$$\vec{E} = -\nabla V-\frac{\partial\vec{A}}{\partial t}$$

$$\left.\begin{aligned}\vec{E} &= -\nabla V-\frac{\partial\vec{A}}{\partial t}\\ \vec{B} &= \nabla\times\vec{A}\end{aligned}\right\}$$

It is possible to set V=0 and still have an electric field via time varying **A**

V and **A** has to change in such a way that E and B remain same.

V, **A** and V', **A'** will have to be related

$$\vec{A}' = \vec{A}+\nabla\lambda$$

$$V' = ?$$

Suppose $\nabla V'+\frac{\partial\vec{A}'}{\partial t} = \nabla V+\frac{\partial\vec{A}}{\partial t}$

$$\nabla(V-V') = \frac{\partial}{\partial t}(\vec{A}-\vec{A}') = -\frac{\partial}{\partial t}\nabla\lambda$$

$$\Rightarrow$$

$$\boxed{\begin{aligned}\vec{A}' &= \vec{A}+\nabla\lambda\\ V' &= V-\frac{\partial\lambda}{\partial t}\end{aligned}}$$

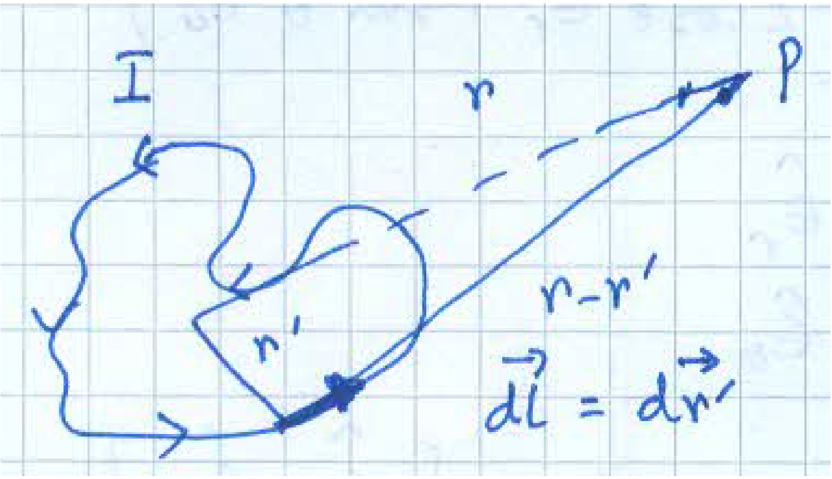
λ is a scalar fn of x, y, z, t

Multipole expansion of the magnetic vector potential

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3\vec{r}\,' = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}}{|\vec{r}-\vec{r}\,'|}$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} (r')^l P_l(\cos\theta)$$

$$(r \gg r')$$



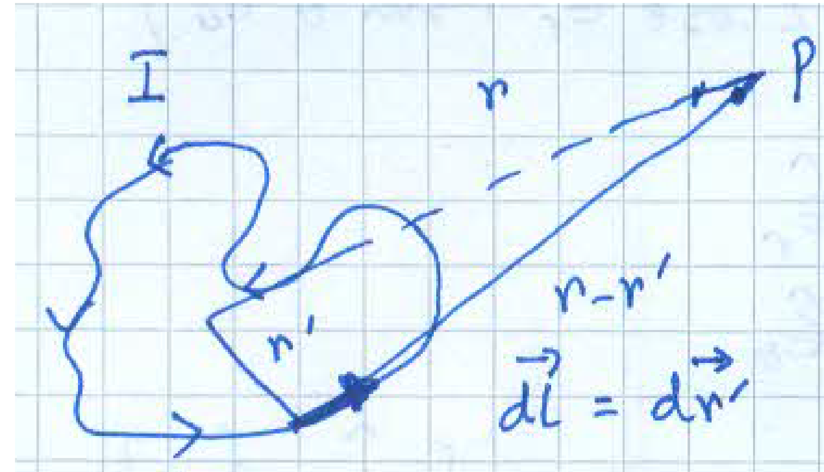
$l=0$ $\quad \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\vec{l}$ Always zero: no magnetic monopoles

$l=1$ $\quad \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\theta\, d\vec{l}$ Origin independent: magnetic dipole

$l=2$ $\quad \frac{\mu_0 I}{4\pi}\frac{1}{r^3}\oint r'^2\frac{1}{2}\left(3\cos^2\theta-1\right) d\vec{l}$ quadrupole

Multipole expansion of the magnetic vector potential

$$\oint r'\cos\theta\, d\vec{r}\,' = -\frac{1}{2}\hat{r}\times\oint \vec{r}\,'\times d\vec{r}\,'$$



$$\text{Hence}\quad \vec{A}_{dipole} = \frac{\mu_0}{4\pi}\left[\frac{1}{2}I\oint \vec{r}\,'\times d\vec{r}\,'\right]\times\frac{\hat{r}}{r^2}$$

$$= \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

$$\frac{1}{2}\oint \vec{r}\,'\times\vec{dl} = \text{area of the loop}$$

$$\text{dipole moment} = \text{current}\times\text{area}$$

In 3D with volume current density

$$\vec{m} = \frac{1}{2}\int \vec{r}\times\vec{J}\,\delta\tau$$

Multipole expansion : an useful identity

For a localised current distribution (J) with zero divergence at all points, and any two scalar functions f,g

$$\int_{vol} \nabla.(\vec{J}\, fg)\, d\tau \;=\; \int_{surf} \vec{J}\, fg.\, d\vec{S} \;=\; 0$$

Since J is localised it vanishes on the surface everywhere.

$$\int_{vol} \left[fg(\nabla.\vec{J}) + \vec{J}.\nabla\, fg \right] d\tau \;=\; 0$$

$$\int_{vol} \left[f\,\vec{J}.\nabla g + g\,\vec{J}.\nabla f \right] d\tau \;=\; 0$$

Take f=1 g=x,y,z in turn & prove

$$\int_{vol} \vec{J}\, d\tau \;=\; 0$$

Take f= g=x,y,z in turn & prove

$$\int_{vol} \vec{r}.\vec{J}\, d\tau \;=\; 0$$

Take f= x, g=y and other permutaions

$$\int_{vol} \left(xJ_y + yJ_x\right) d\tau \;=\; 0$$

Multipole expansion of the magnetic vector potential

$$A_i(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{J_i(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3\vec{r}\,'$$

$$= \frac{\mu_0}{4\pi}\left[\frac{1}{r}\underbrace{\int J_i(\vec{r}\,')d^3\vec{r}\,'}_{=0} + \frac{1}{r^3}\vec{r}.\int \vec{r}\,' J_i(\vec{r}\,')d^3\vec{r}\,' + \cdots\right]$$

$$= \frac{\mu_0}{4\pi r^3}\vec{r}.\int \vec{r}\,' J_i(\vec{r}\,')d^3\vec{r}\,' + \cdots$$

Need to use the identities derived just before to obtain the result in the next step

$$= \frac{\mu_0}{4\pi r^3}\left[-\frac{1}{2}\vec{r}\times\int \vec{r}\,'\times\vec{J}\,d^3\vec{r}\,'\right]_i + \cdots$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3} + \cdots$$

The magnetic vector potential and field of a perfect dipole

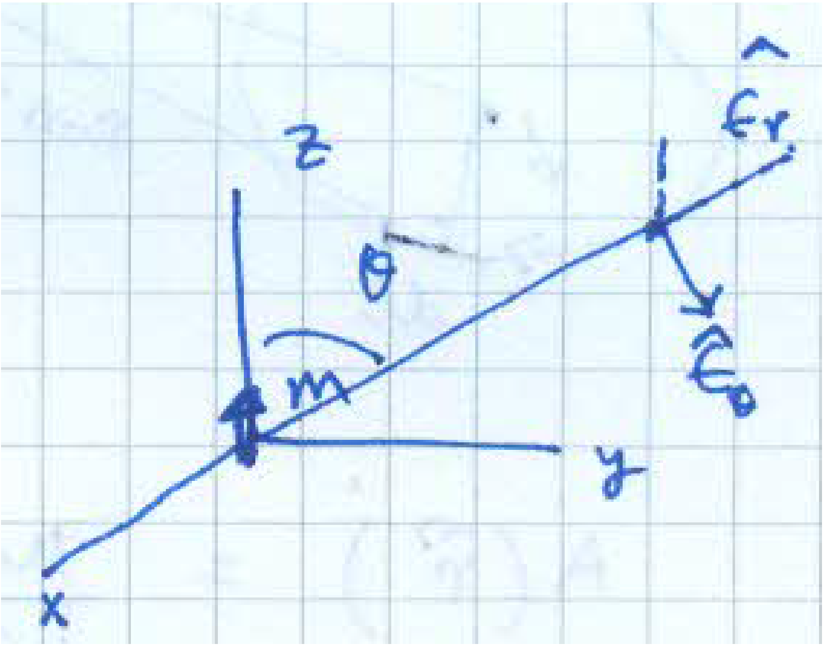
$$\vec{A}_{dipole} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\epsilon}_\phi$$

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}\left(2\cos\theta\,\hat{\epsilon}_r + \sin\theta\,\hat{\epsilon}_\theta\right)$$

$$Since \begin{cases} m\cos\theta = \vec{m}.\hat{\epsilon}_r \\ m\sin\theta = \vec{m}.\hat{\epsilon}_\theta \end{cases}$$

$\vec{m}$ lies in the plane defined by $[\epsilon_r, \epsilon_\theta]$

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}.\hat{r})\hat{r} - \vec{m}}{r^3}\right] \qquad (r \neq 0)$$



$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}.\hat{r})\hat{r} - \vec{m}}{r^3}\right] + \mu_0\frac{2}{3}\vec{m}\,\delta(\vec{r})}$$

If the r=0 point is to be correctly handled then the delta fn is needed.

Correct solution of the interaction of electron spin and nuclear spin (hyperfine) requires this.

Forces and torques on current loops and dipoles

Force on a current distribution will also vanish if all the current loops are closed and the fields are constant

$$\vec{F} = q\left(\vec{E}+\vec{v}\times\vec{B}\right) \qquad (single\ particle)$$
$$\delta\vec{F} = (n\,\delta\tau)\,q\left(\vec{E}+\vec{v}\times\vec{B}\right) \qquad (many\ particles)$$
$$\delta\vec{F} = \vec{J}\times\vec{B}\,\delta\tau = I\,\delta\vec{l}\times\vec{B} \qquad (current\ line\,,\ distrib)$$

A current carrying wire is electrically neutral because it always has equal number of electrons and positive ions in lattice. An electric field does not create a net force on it.

Magnetic field does, becuase the electrons are moving and the fixed ions in the lattice are not – so the lattice sees no Lorentz force.

Useful facts to remember....

Also $\nabla\cdot\vec{J} = 0,$ since $\dfrac{\partial\rho}{\partial t}=0$

$$\oiiint \vec{J}\,d\tau = 0$$

(for a localised charge distribution)... *why?*

Consider the expression

$$\int_{vol}\nabla\cdot(x\vec{J})\,d\tau = \int\left[x\nabla\cdot\vec{J}+\vec{J}\cdot\nabla x\right]d\tau$$

But $\vec{J}\cdot\nabla x = J_x$

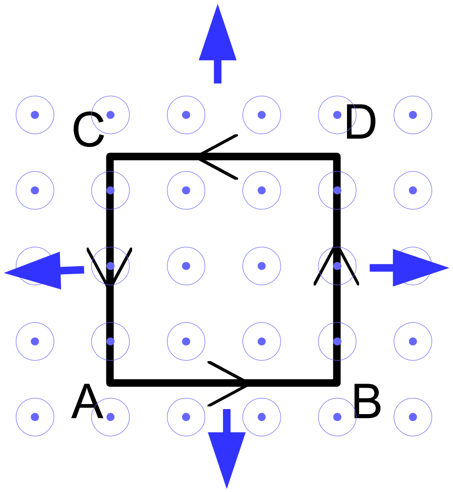
Also $\vec{J} = 0$

on a large bounding surface

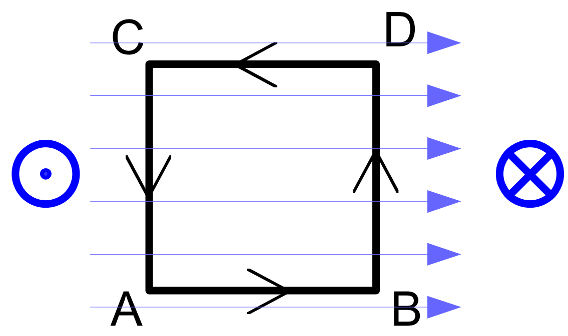
So the result follows

Forces and torques on current loops and dipoles

What is the force and torque on the square loop?



F=0
Torque = 0

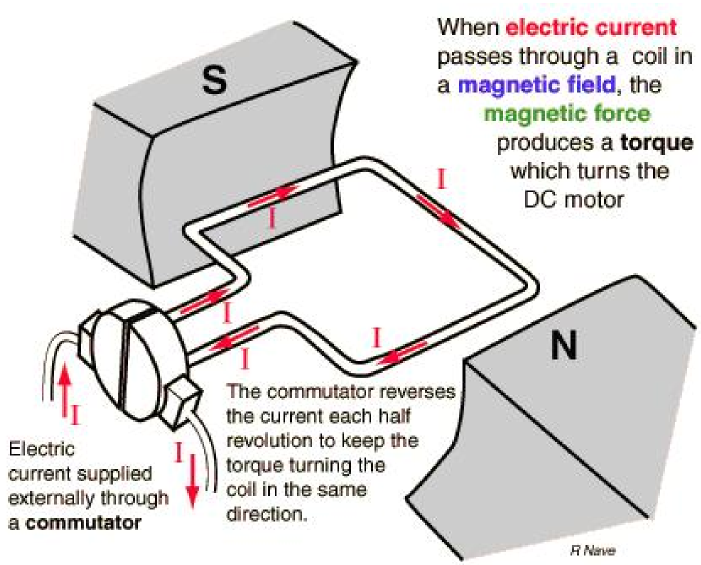


$$\vec{F} = \oint I \,\delta \vec{l} \times \vec{B}$$

Total F=0, still, since opposite sides have exactly opposite dl vector

Torque = BI x area (in magnitude)

The basis for electric motor winding, pointer type current measuring meters etc.



Only inhomogeneous magnetic field can create a force on a current loop/ dipole.

Torque is possible with uniform fields.

Forces and torques on current loops and dipoles

Consider an arbitrary current distribution in a spatially varying field **B**.
Question: What is the force and torque on it?
Assume that the current density is confined to a small volume.

$$\vec{F} = \int \left(\vec{J}\times\vec{B}\right) d^3\vec{r}\,' \quad \text{and} \quad B_k(\vec{r}) = B_k(0) + \vec{r}.\nabla B_k$$

$$F_i = \epsilon_{ijk}\Big[\underbrace{B_k(0)\int J_j(\vec{r}\,')\, d^3\vec{r}\,'}_{=0} + \underbrace{\int J_j(\vec{r}\,')\vec{r}\,'.\nabla' B_k(0)\, d^3\vec{r}\,'}_{\text{Field inhomogeneity giving rise to force}} + \ldots\Big]$$

We need to simplify: $\nabla B_k(0).\int \vec{r}\,' J_j(\vec{r}\,') d^3\vec{r}\,'$

This will give : $-\left[\nabla B_k(0)\times \underbrace{\frac{1}{2}\int (\vec{r}\,'\times\vec{J}) d^3\vec{r}\,'}_{\text{Dipole moment of the current distribution}}\right]_i$

The proof is similar to the one given for the dipole moment calculation before.

$$\vec{F} = (\vec{m}\times\nabla)\times\vec{B} = \nabla(\vec{m}.\vec{B}) - \vec{m}\underbrace{(\nabla.\vec{B})}_{=0}$$

Forces and torques on current loops and dipoles

The torque will be given by

$$\begin{aligned}\vec{\tau} &= \int \left[\vec{r}\,'\times\left(\vec{J}\times\vec{B}\right)\right]d^3\vec{r}\,' \\ &= \int\left[\vec{J}\left(\vec{r}\,'.\vec{B}\right) - \vec{B}\left(\vec{r}\,'.\vec{J}\right)\right]d^3\vec{r}\,' \\ &= \vec{B}.\int \vec{r}\,'\vec{J}\,d^3\vec{r}\,' + \underbrace{\vec{B}.\int\left(\vec{r}\,'.\vec{J}\right)d^3\vec{r}\,'}_{=0} \\ &= \vec{m}\times\vec{B}\end{aligned}$$

Since the zeroth order term does not vanish, we take the value of B at a fixed point in the distribution and treat it as a constant

Electric dipole

$$\begin{aligned}\vec{F} &= \nabla\left(\vec{p}\,.\vec{E}\right) \\ \vec{\tau} &= \vec{p}\times\vec{E}\end{aligned}$$

Magnetic dipole

$$\begin{aligned}\vec{F} &= \nabla\left(\vec{m}\,.\vec{B}\right) \\ \vec{\tau} &= \vec{m}\times\vec{B}\end{aligned}$$

The end result is very similar, though the internal mechanism is quite different.

How does matter acquire magnetic polarisation (dipole moment)

The mechansims by which matter acquires a magnetic "dipole moment" per unit volume is more complex than the way electric polarisation is acquired.

A classical description of this is not really possible.

The magentic moment acquired may be

1. In the direction of the magnetic field but very weak. (paramagnetism)
2. OPPOSITE to the direction of the applied field and also very weak (diamagnetism)

   Para & dia magnetic effects disappear when the applied field is removed.

3. In the direction of the applied field but very strong and remains even after the initial field is removed .(ferromagnetism)
   This is characterised by hysteresis effects/loops/

These effects involve the dynamics of orbital electrons of an atom/ free electrons in a metal in a magnetic field, which requires a quantum mechanical description.

We will not focus on "how" the polarisation is acquired.

Magnetic polarisation and its description

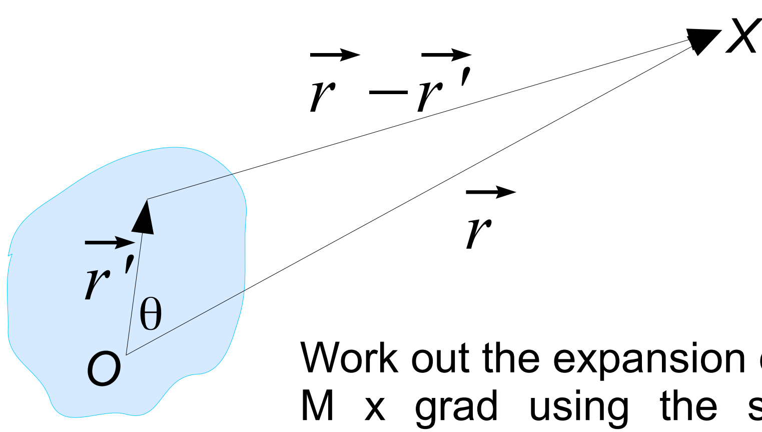
$$\vec{A} = \frac{\mu_0}{4\pi}\int_{vol} \frac{\vec{M}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} d^3\vec{r}\,'$$

Recall that

$$\nabla_{r'} \frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

Hence

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{vol} \vec{M}(\vec{r}\,')\times\nabla_{r'} \frac{1}{|\vec{r}-\vec{r}\,'|} d^3\vec{r}\,'$$



Work out the expansion of M x grad using the standard rules. Convert one volume integral of a curl to a surface integral.....

M is the magnetic moment per unit volume.

The unit of **M** can be defined in two ways:

[m] = current x area (so Ampere .m²)
[M] = Am^{-1}

Also [m.B] = energy, hence
[m] = Joule/Tesla
[M] = J/Tesla/m^{-3}

But historically an unit emu has been used. emu =1erg/gauss

From which we can define emu/cc or emu/gm

1 erg/gauss = 10^{-3} Ampere .m²

Atomic magnetic moments are of the order of a "Bohr magneton“

Magnetic polarisation and its description

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{vol} M(\vec{r}\,')\times\nabla_{r'}\frac{1}{|\vec{r}-\vec{r}\,'|}d^3\vec{r}\,'$$

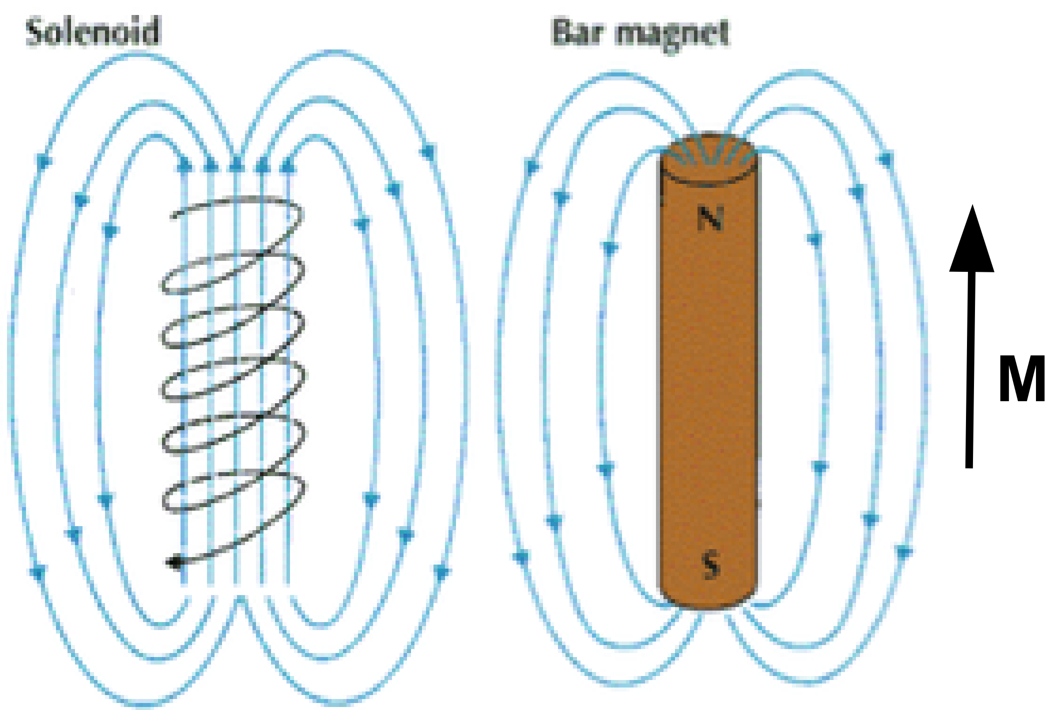
$$= \frac{\mu_0}{4\pi}\left[\int_{vol}\frac{1}{|\vec{r}-\vec{r}\,'|}(\nabla\times\vec{M})\,d\tau + \int_{surf}\frac{1}{|\vec{r}-\vec{r}\,'|}\vec{M}\times d\vec{S}\right]$$

$$\vec{J}_b = \nabla\times\vec{M} \qquad \& \qquad \sigma_b = \vec{M}\times\hat{n}$$

need to use the relation

$$\int_{vol}\nabla\times\vec{A}\,d\tau = -\int_{surf}\vec{A}\times d\vec{S}$$

Contribution from a volume current and a surface current density.



$$\vec{M} = constant$$

$$\nabla\times\vec{M} = 0$$

$$\vec{M}\times\hat{n} = M\,\hat{\epsilon}_\phi$$

mimics the solenoid current

equivalnet ampere turns per mt

Magnetic polarisation and its description: **B H M** vectors

$$\nabla \times \vec{B} = \mu_0 \vec{J} = \mu_0 \underbrace{(\vec{J}_f + \vec{J}_b)}$$

"Free" current put in by wires, solenoids etc.

"Bound" current due to induced or frozen magnetic dipoles

$$\vec{J}_b = \nabla \times \vec{M} \quad \textit{hence}$$

$$\nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_f$$

$$\textit{call} \quad \frac{\vec{B}}{\mu_0} - \vec{M} = \vec{H}$$

$$\boxed{\begin{aligned} \nabla \times \vec{H} &= \vec{J}_f \\ \nabla . \vec{H} &= ? \end{aligned}}$$

!! NOT NECESSARILY ZERO!!

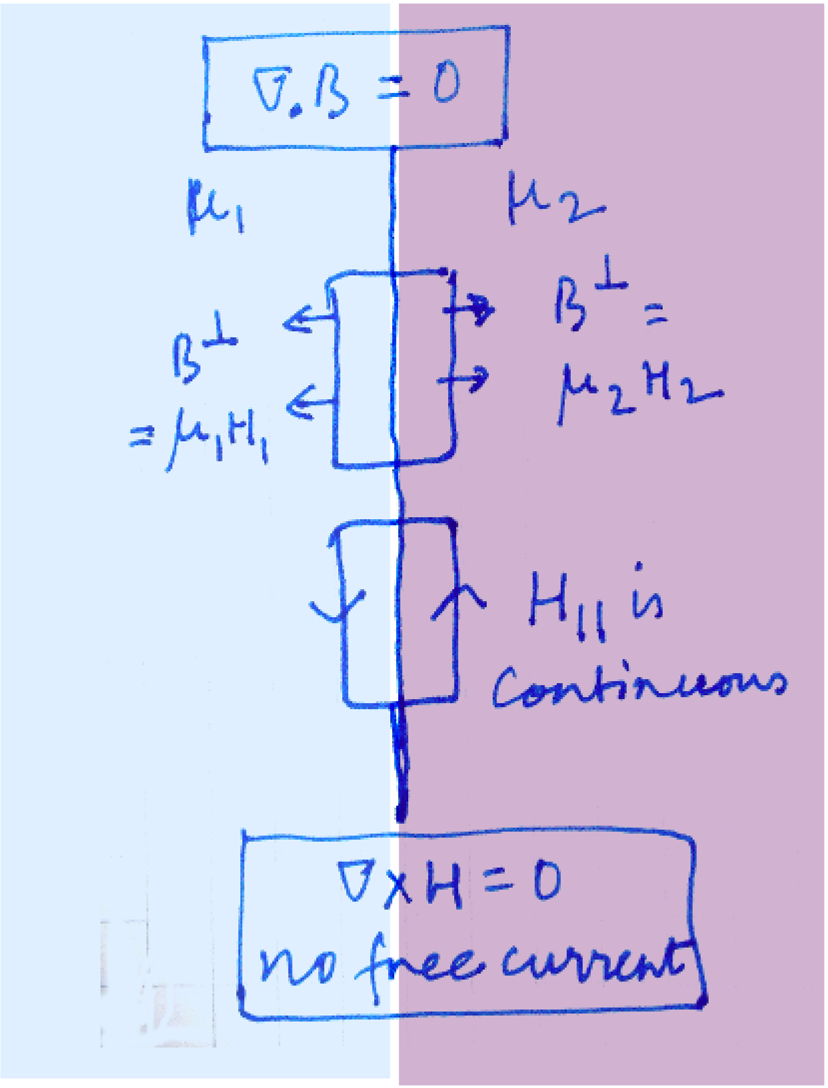
Historically a proportionality between **M** and **H** was emphasized as a material property. This leads to:

$$\begin{aligned} \vec{B} &= \mu_0 (\vec{H} + \vec{M}) \\ \vec{M} &= \chi \vec{H} \\ \vec{B} &= \mu_0 (1 + \chi) \vec{H} \\ \vec{B} &= \mu \vec{H} \end{aligned}$$

χ is called susceptibility

μ is called permeability

Magnetic polarisation and its description: **B H M** vectors : Boundary conditions



$$\vec{J}_{free} = 0 \qquad (at \quad the \quad interface)$$

$$\mu_1 H_1^{\perp} = \mu_2 H_2^{\perp} \qquad (since\, \nabla . \vec{B} = 0)$$

$$H_1^{\parallel} = H_2^{\parallel} \qquad (since\, \nabla \times \vec{H} = 0)$$

The unit of H is same as the unit of M.
In SI ampere-turn per meter is generally used. It is dimensionally different from Tesla.

In cgs unit of B and H have same dimensionality.
Gauss is used for B
Oersted is used for H

Confusion is very common between B and H

Divergence of H is not necessarily zero : An example

Consider a bar magnet with magnetisation M

$$Air : \vec{H} = \frac{\vec{B}}{\mu_0} \quad (M = 0)$$

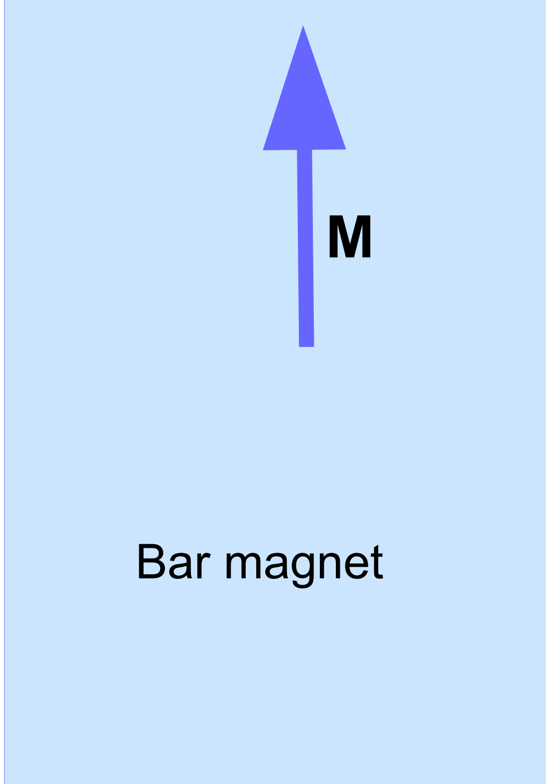
$$B_{\text{air}}^{\perp} = B_{\text{bar}}^{\perp}$$

$$H_{\text{air}}^{\perp} \neq H_{\text{bar}}^{\perp}$$

$$\nabla \times \vec{H} = 0$$
$$\nabla . \vec{H} = -\nabla . \vec{M} = \rho_m$$

is sometimes useful to describe an assembly of magnets via a potential ϕ such that $\vec{H} = -\nabla \phi$

$$\text{Bar} : \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$



So H can have "sources" and "sinks" like the electric field.

In cases where curl **H** = 0, it is possible to construct a magnetic scalar potential, whose gradient would give **H**.

In older texts "magnetic pole density" etc are used. These are the sources and sinks of H, like electric charge is the source and sink of E.

This leads to some confusion about H being the "real" field, which is wrong!!

Physical interpretation of the bound currents

Field of an uniformly magnetised sphere

$$\vec{J}_b = \nabla \times M = 0$$

$$\vec{\sigma}_b = \vec{M} \times \hat{n} = M \sin\theta \, \epsilon_\phi$$

Integrate directly to find $\vec{A}$ and then $\vec{B}$

$$\vec{A} = \frac{\mu_0 M R^3}{3} \frac{\sin\theta}{r^2} \hat{\epsilon}_\phi \qquad (r > R)$$

$$\vec{A} = \frac{\mu_0 M}{3} r \sin\theta \, \hat{\epsilon}_\phi \qquad (r < R)$$

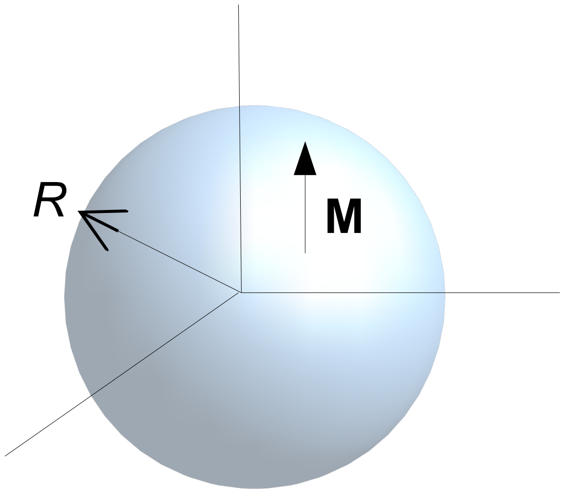
$$\boxed{\begin{aligned} \vec{B} &= \frac{\mu_0}{4\pi}\left(\frac{4\pi R^3 M}{3}\right)\left(\frac{2\cos\theta\,\hat{\epsilon}_r + \sin\theta\,\hat{\epsilon}_\theta}{r^3}\right) \\ \vec{B} &= \frac{2\mu_0}{3}\vec{M} \end{aligned}}$$

Inside the sphere

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = -\frac{\vec{M}}{3}$$

directed opposite to $\vec{B}$ and $\vec{M}$

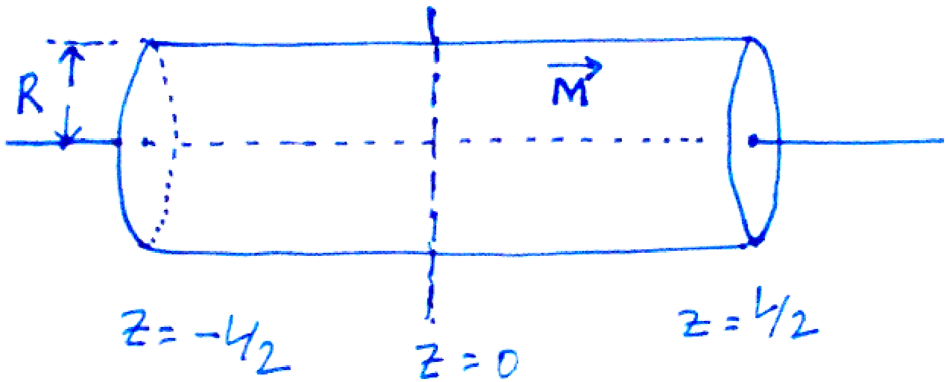
A somewhat counter-intuitive result!



Field inside is constant

Field outside is that of an equivalent dipole placed at origin.

Field of a cylindrical bar magnet



Replace the Magnetisation by an equivalent current: M=NI and calculate the axial field due to all the current loops, using the result for a single loop

$$B(0,0,z) = \frac{\mu_0 M}{2}\left[\frac{z+L/2}{\sqrt{R^2+(z+L/2)^2}} - \frac{z-L/2}{\sqrt{R^2+(z-L/2)^2}}\right]$$

$$\text{Field just outside} \quad : \quad B \approx \frac{\mu_0 M}{2} \qquad (z=\pm L/2)$$

$$\text{Field very far away} \quad : \quad B \approx \frac{\mu_0}{2\pi z^3}.(M\pi R^2 L) \qquad (z \gg L)$$

Inverse cube fall off of a dipole field

Calculate H and show that it points opposite to M inside the bar.
Typical strong permanent magnets have remnance $\mu_0 M \sim 1\, Tesla$

Typical values of susceptibilities

TABLE 6.1 MAGNETIC SUSCEPTIBILITIES

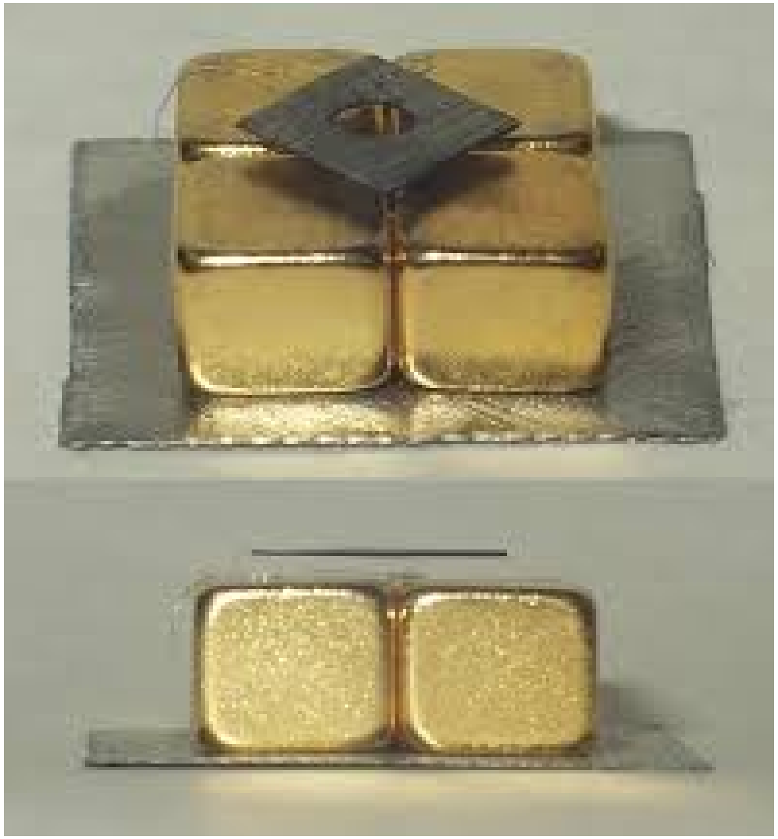
Material	Magnetic Susceptibility
Diamagnetic:	
Bismuth	-16.5×10^{-5}
Gold	-3.0×10^{-5}
Silver	-2.4×10^{-5}
Copper	-0.96×10^{-5}
Water	-0.90×10^{-5}
Carbon Dioxide	-1.2×10^{-8}
Hydrogen	-0.22×10^{-8}
Paramagnetic:	
Oxygen	190×10^{-8}
Sodium	0.85×10^{-5}
Aluminum	2.1×10^{-5}
Tungsten	7.8×10^{-5}
Gadolinium	$48{,}000 \times 10^{-5}$

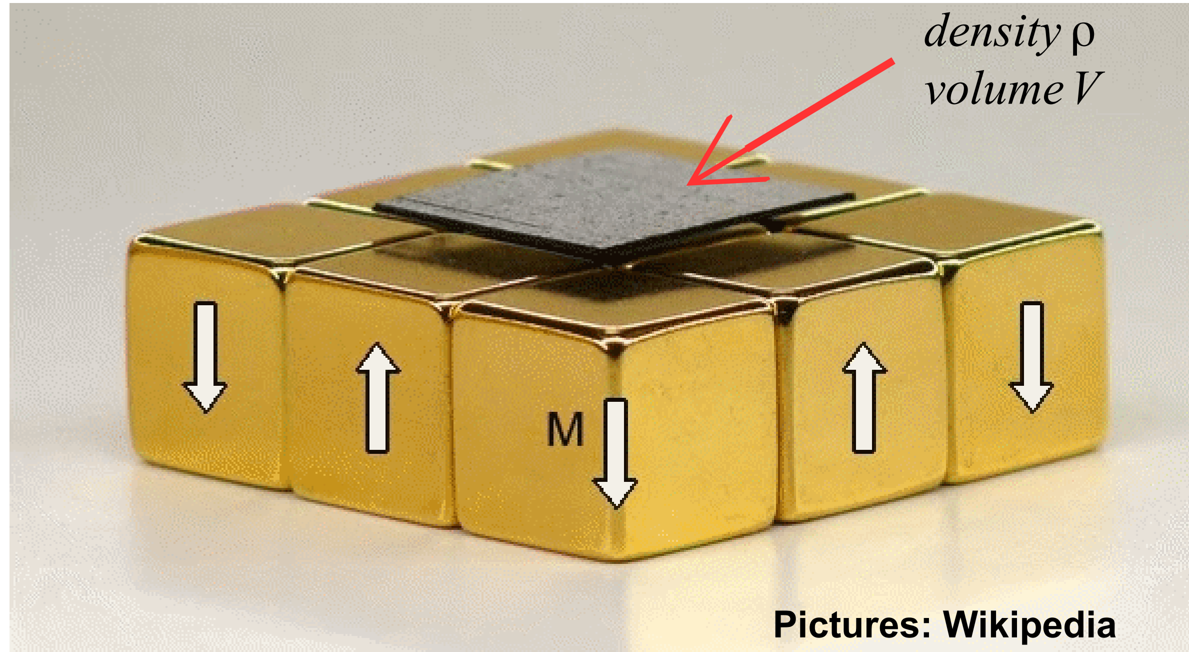
Source: Handbook of Chemistry and Physics, 67th ed. (Cleveland: CRC Press, Inc., 1986–87.) All figures are for atmospheric pressure and room temperature.

Graphite's suscpetibility can be -6e-4 to -1e-5 depending on orientation in SI units

Stability of levitation: Why does it require diamagnets ?

Since magnetic field repels "diamagnets", it can be made to float in a region of strongly varying (large gradient) magnetic field.





Pieces of graphite floating on a strong magnets. The magnets are typically 5-10mm cubes and would have a remnance of 1-2 Tesla.

The height at which these float are typically 1-2 mm

$$\vec{F} = \nabla(\vec{m}.\vec{B}) = \rho V g \,, \quad where \quad \vec{m} \approx V \frac{\chi}{\mu_0} \vec{B}$$

Stability requires that $\nabla.(\nabla \chi \vec{B}.\vec{B}) < 0$

$$But \quad \nabla^2 \vec{B}.\vec{B} = \nabla^2 (B_x^2 + B_y^2 + B_z^2)$$

$$Show\, that \; : \; \sum_{ij} \frac{\partial^2}{\partial x_i^2} B_j B_j = 2 \sum_j |\nabla B_j|^2$$

$$Hence \quad \nabla^2 \vec{B}.\vec{B} > 0 \quad always$$

stability requires $\nabla.\vec{F} < 0$ possible only if $\chi < 0$

Electrodynamics

Something needs to be added to Ampere's Law. Why?
Can we decouple E and B?
Emergence of an wave equation. Why is f(x-vt) a "wave"?
How does the displacement current term compare with normal current?

Induced emfs: Inductors and generators

Lorentz force Law in potential form (convective derivative)
Energy and momentum of the EM field.

Maxwell's equation in matter
Refractive index
Reflection and transmission of em waves at an interface

Why there must be something more in Ampere's Law.

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

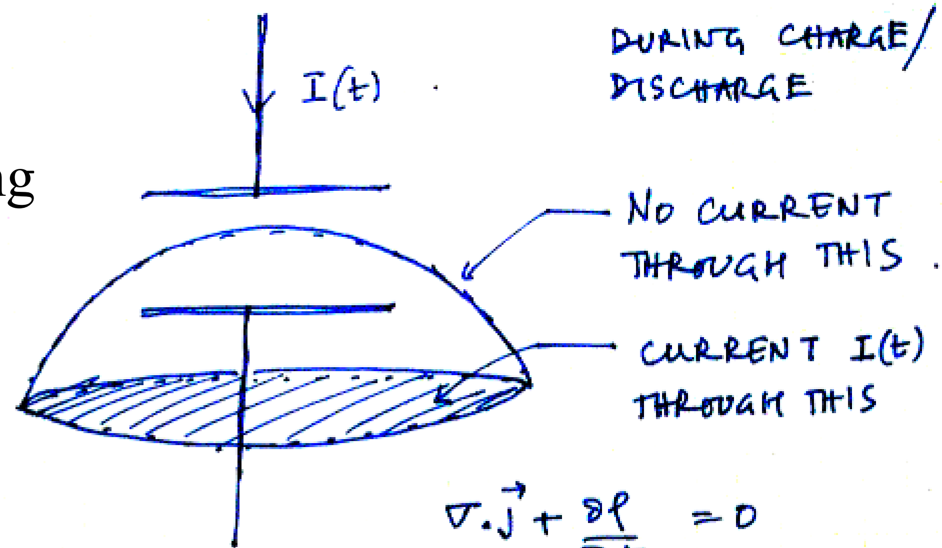
If $\nabla . \vec{J} = 0$ then it is fine

Cannot be correct when $\vec{J}$ is changing

$$\nabla . \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\nabla . \vec{E} = \frac{\rho}{\epsilon_0}$$

always true



$$\nabla . \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\nabla . J \neq 0 .$$

Replace RHS by

$$\vec{J} \rightarrow \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Question: How does E vary within the plates

$$E = \frac{V}{d} = \frac{Q}{Cd}$$ Q: Charge on the plates

$$\frac{\partial E}{\partial t} \propto \frac{\partial Q}{\partial t} \propto I(t)$$

Full time dependent equation becomes

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

The correction term should have something to do with $\frac{\partial E}{\partial t}$.

Historically the additional term is called "displacement current"

The full set of Maxwell's equations

electrostatics

$$\nabla . \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Faraday Induction

$$\nabla . \vec{B} = 0$$

magnetostatics

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell's displacement current term

These are first order differential equations.
Decoupling them would invariably lead to second order equations.

First do it for free space where there are no charges and currents

Decoupling of Maxwell's equations leads to wave equation in free space

$$\nabla \times B = \epsilon_0 \mu_0 \frac{\partial E}{\partial t}.$$

$$\nabla \times \nabla \times B = \epsilon_0 \mu_0 \frac{\partial}{\partial t}(\nabla \times E)$$

$$\nabla(\nabla \cdot B) - \nabla^2 B = \epsilon_0 \mu_0 \frac{\partial}{\partial t}\left(-\frac{\partial B}{\partial t}\right)$$

$$\nabla^2 B - \epsilon_0 \mu_0 \frac{\partial^2 B}{\partial t^2} = 0$$

$$\boxed{\epsilon_0 \mu_0 = \frac{1}{c^2}}$$

The velocity of light emerges "naturally" from Maxwell's equations

$$\nabla \times E = -\frac{\partial B}{\partial t}.$$

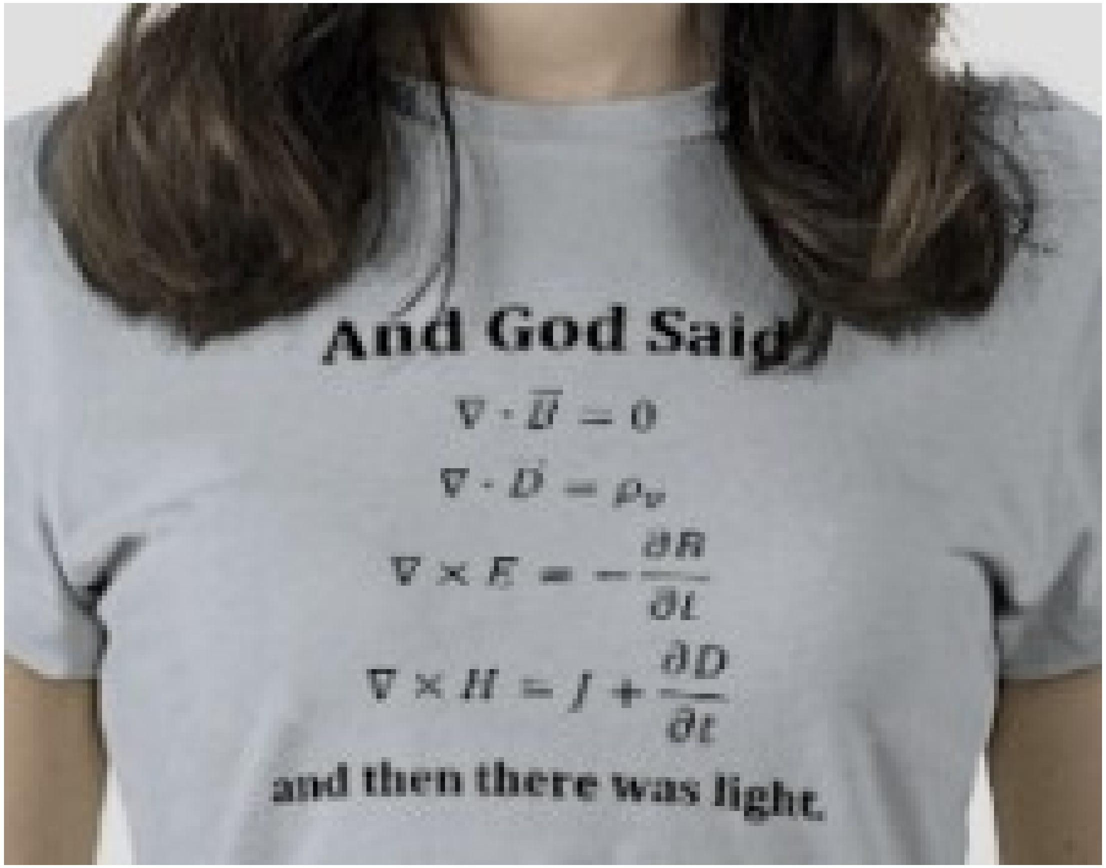
$$\nabla \times \nabla \times E = -\frac{\partial}{\partial t}(\nabla \times B)$$

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\frac{\partial}{\partial t}\left(\epsilon_0 \mu_0 \frac{\partial E}{\partial t}\right)$$

$$\nabla^2 E - \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} = 0$$

What is the generic solution of the wave equation?

Decoupling of Maxwell's equations leads to wave equation in free space



When does the displacement current term become more dominant?

Inductors and generators : Self Inductance

A current carrying loop creates a magnetic field.
Some of the magnetic field creates a flux through the loop.

$$\Phi = LI \qquad \text{similar to} \qquad V = Q/C$$

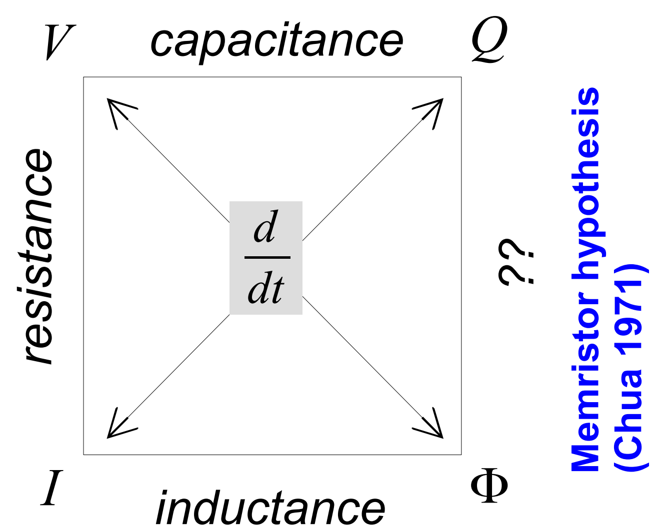
How much energy does it take to set this up?

$$\begin{aligned} V &= -L\frac{dI}{dt} \qquad (Faraday \quad induction) \\ \delta W &= V\,\delta Q \qquad (work\ done\ by\ the\ battery) \\ \frac{dW}{dt} &= VI = \left(\frac{d}{dt}LI\right)I \\ &= \frac{d}{dt}\left(\frac{LI^2}{2}\right) \end{aligned}$$

Consider a coil of *N* turns,
With length *l* and cross sectional area *A*

$$\begin{aligned} B &= \mu_0 \frac{N}{l} I \\ LI &= NBA \\ \frac{(LI)\,I}{2} &= \frac{NBA}{2}\,\frac{Bl}{\mu_0 N} \\ W &= \frac{B^2}{2\mu_0} Al \end{aligned}$$

Energy per unit volume in magnetic field



In a circuit capacitor stores energy in its electric field. The inductor stores energy in its magnetic field.
More generally...

Energy in the electromagnetic field

$$\int_{vol} d\tau \left[\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right]$$

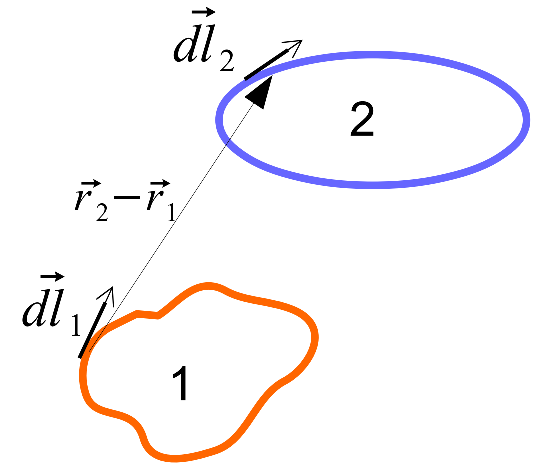
Flux through loop 2 due to loop 1

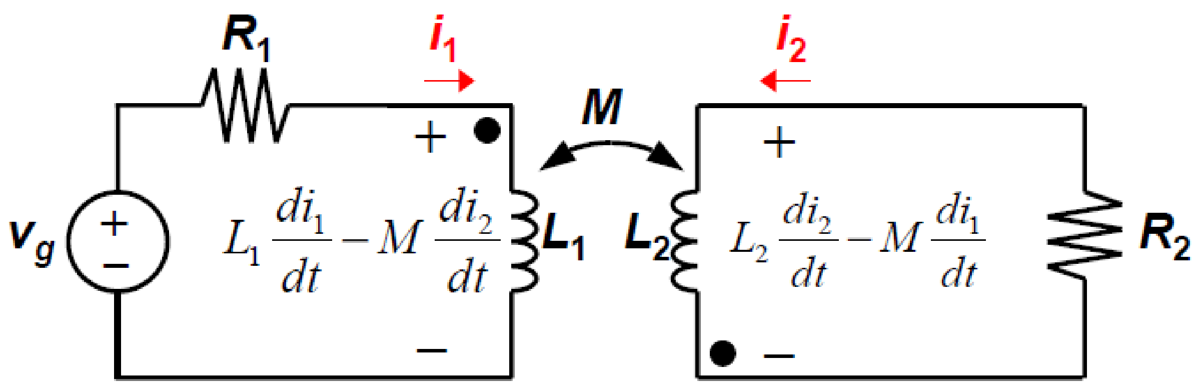
$$\Phi_2 = \int \vec{B}_1 . d\vec{S}_2 = \int (\nabla \times \vec{A}_1) . d\vec{S}_2$$

$$= \int \vec{A}_1 . d\vec{l}_2 = \int \frac{I_1 d\vec{l}_1}{|\vec{r}_1 - \vec{r}_2|} . \int d\vec{l}_2$$

$$M\, I_1 = I_1 \oint \oint \frac{d\vec{l}_1 . d\vec{l}_2}{|\vec{r}_1 - \vec{r}_2|}$$

Mutual Inductance
(notice the symmetry)





Question :

$$\Phi_1 = LI_1 + MI_2$$

OR

$$\Phi_1 = LI_1 - MI_2$$

The flux due to coil 2 may increase or decrease the flux through coil 1
That depends on the geometry and assumed direction of the instantaneous current

The dot convention : If both currents are entering the dot then fluxes will add.

Equivalently : If current enters the dotted terminal then the polarity of the voltage at the other dotted terminal will be positive.

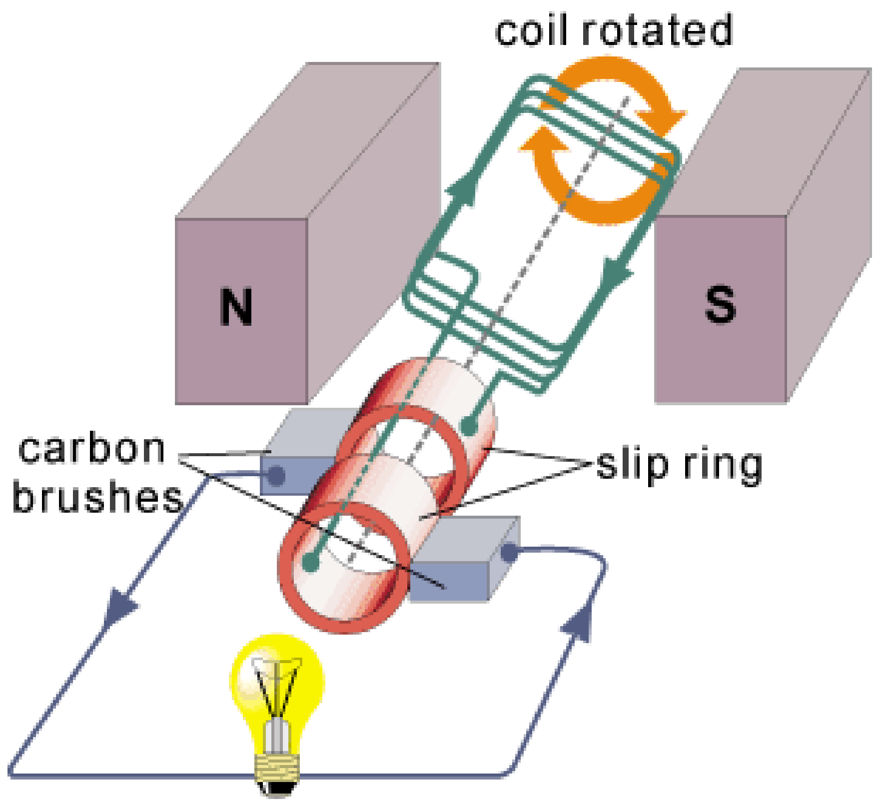
How much is the energy stored in the system L1 + L2 ?

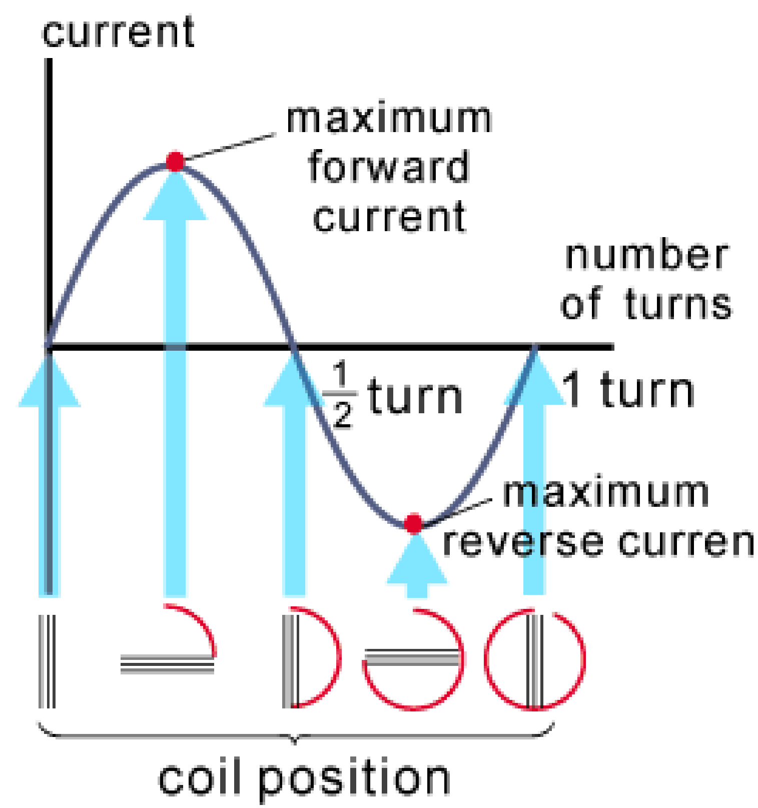
$W = L_1 I_1^2 + L_2 I_2^2 + M I_1 I_2$ When all currents either enter or leave the dot

$W = L_1 I_1^2 + L_2 I_2^2 - M I_1 I_2$ When one current enters and the other leaves the dot

This follows from the relation between dot and the addition/subtraction of flux
Can be generalised to an arbitrary number of coupled linear inductors

Inductors and generators : The simple generator





http://misswise.weebly.com/motors-and-generators.html

Either the magnet OR coil may rotate

Notice that the voltage WILL alternate or have ripples even if the contacts to the terminals are flipped every half cycle

If the magnet rotates, the change in flux is obvious

$$\frac{d}{dt}\Phi = \frac{d}{dt} N.B.A.cos\,\omega\, t$$

$$V = -N.B.A.\,\omega \sin \omega\, t$$

N turns in the coil
Area of each turn : A
angular freq of rotation : ω

Inductors and generators : The simple generator : coil rotates, magnet stationary

$$\oint_{coil} \vec{E}\,.\,\vec{dl} \;=\; \oint_{coil} \vec{v}\times\vec{B}\,.\,\vec{dr} \qquad (Lorenz\ Force)$$

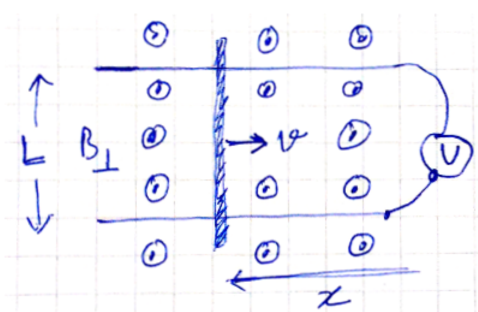
Notice the **B** vector is fixed in this case

For rigid rotation : $\vec{v}=\vec{\omega}\times\vec{r}$ $\quad so \quad$ $(\vec{\omega}\times\vec{r})\times\vec{B} \;=\; \vec{r}(\vec{B}\,.\,\vec{\omega}) \;-\; \vec{\omega}(\vec{B}\,.\,\vec{r})$

$\vec{B}\,.\,\vec{\omega} \;=\; 0$ $\quad$ & $\quad$ $\vec{B}\,.\,\vec{r} \;=\; Br\sin\omega t$ $\quad$ ($\theta=\omega t$ is the angle with the normal to the coil)

The integral again gives $V \;=\; -B.A.\omega\,.\sin\omega t$

Example: Conducting rod rolling on a track.....(Lorenz force or flux change??)



FLUX

$$\Phi = BLx\,.$$

$$\frac{d\Phi}{dt} = BL\frac{dx}{dt}\,.$$

LORENTZ FORCE

$$F = qBv$$

$$\int_0^L \frac{\vec{F}}{q}\,.\,dl = BvL$$

So "flux changes due to magnet motion" and "circuit moves" cases gives the same result. [See refs for interesting comments]

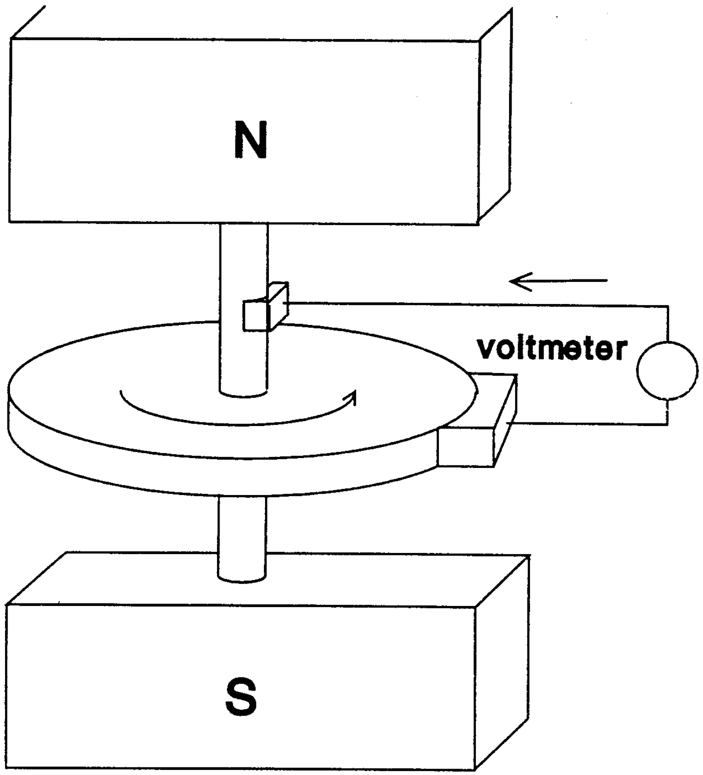
The usual viewpoint is : Faraday induction is correct in the frame where circuit is at rest. But this is *not* always easy to apply...becuase the circuit may deform, expand or stretch (motion, but not centre of mass motion.)

EITHER apply Faraday's law, treating the derivative carefully
OR apply Lorentz Force argument

References:

Feynman Lect vol2, chap16
J D Jackson Chap 6

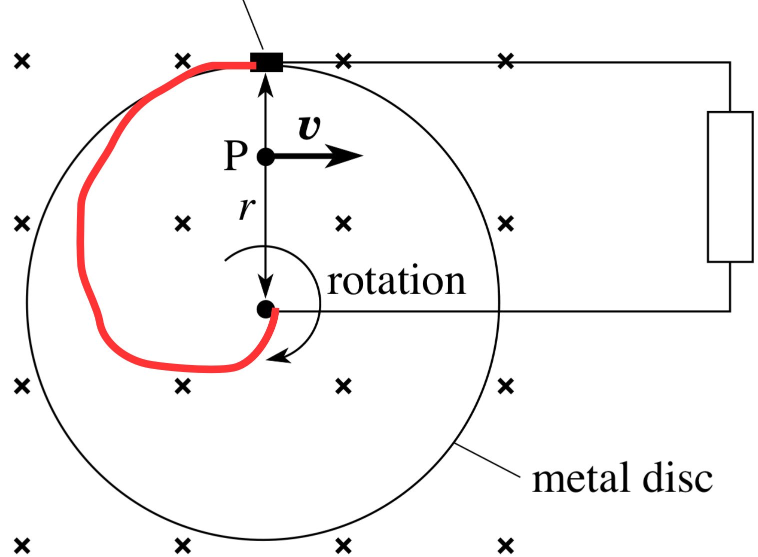
Inductors and generators : The NOT SO SIMPLE homopolar generator



Faraday disk/ homopolar/unipolar generator.
Question: Where is the change of flux?
Why does it generate a voltage at all?

Observation: Rotating the disk generates a voltage.
But rotating the magnet does NOT.

How is the "circuit" being completed?
What is the path of the electron?



uniform magnetic field ***B*** acting into the page

Force on the moving charge : $\vec{F} = q\,\vec{v}\times\vec{B}$

In travelling from center to edge of the disk

$$\int \frac{\vec{F}}{q}\cdot d\vec{l} = \int_0^r \vec{v}\times\vec{B}\cdot d\vec{l}$$

$$= B\,\omega\,\frac{r^2}{2}$$

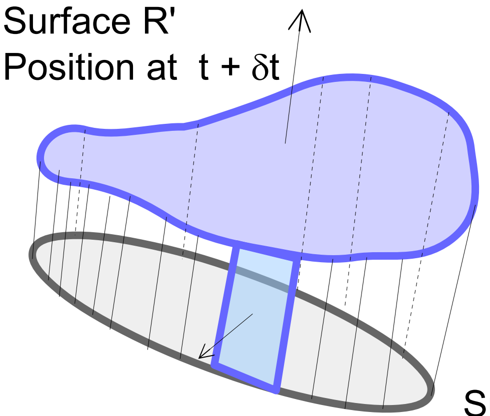
$$\vec{B} = B\,\hat{\epsilon}_z$$

$$\vec{v} = \omega\, r\,\hat{\epsilon}_\theta$$

How does someone sitting in the disk explain the voltage generated?

Faraday induction vs Lorenz force. Are they equivalent?

circuit is moving through $\vec{B}(x,y,z)$
which has no t dependence
Loop/circuit may not be rigid...
parts may have diff speeds



$$\vec{da} = \vec{dl} \times \vec{v}\,\delta t$$

Since $\vec{B}$ has no t dependence

$$\oiint_{R+S+R'} \vec{B}.\vec{da} = 0 \qquad \textit{always / any closed surface}$$

When R is considered part of the closed surface the direction of the outward normal reverses. So the sign also revereses. Important!

$$\Phi(t+\delta t) - \Phi(t) = \oiint_{R'} \vec{B}.\vec{da} + \oiint_{R} \vec{B}.\vec{da}$$

$$\delta\Phi = -\oiint_{S} \vec{B}.d\,\vec{a}$$

$$= -\oint_{R} \vec{B}.\vec{dl} \times \vec{v}\,\delta t$$

$$-\frac{d\Phi}{dt} = \oint_{R} \vec{v} \times \vec{B}.\vec{dl}$$

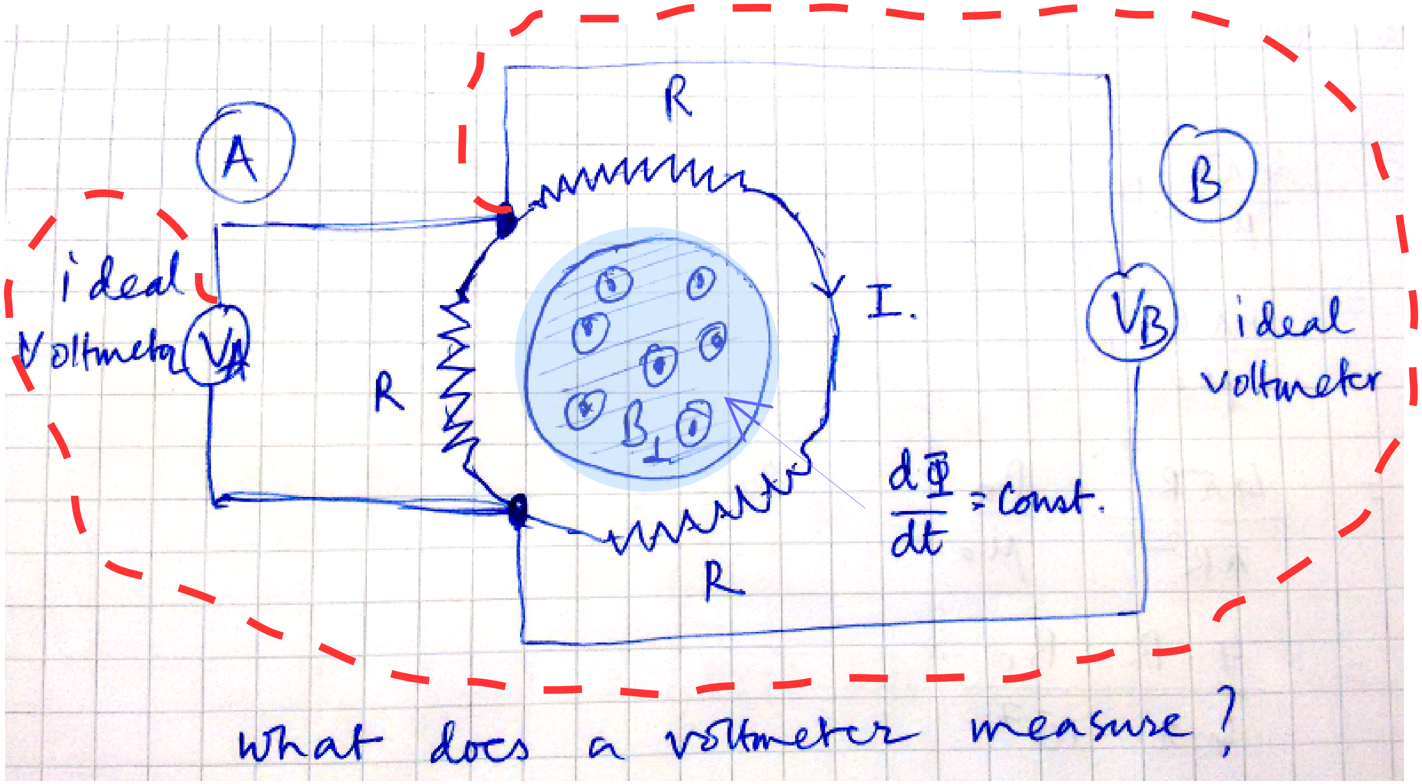
But this will not work if B has explicit time dependence because the surfaces R and R' are not traced out at equal time, we cannot set integral of B.da to zero over the surface.

But Faraday's law is true irrespective of whether B is time dependent or not.

The two laws are NOT equivalent, but of course consistent with each other!

"Voltage" in a non-conservative field situation

Non conservative fields (non zero curl) can give rise to puzzling situations. An example: flux changes in the central region and induced a current in the R+R+R loop.



Two voltmeters are connected across the <u>same two points</u>.
Q: Should they not measure the same "voltage"?
A: NO, not in this case.

But....if the top lead of A is disconnected and laid along the red dotted line, then both will measure the same. (figure out why..)

What do "voltmeters" measure? Faraday's law in a multiply connected region.
R H Romer, *American Jl of Physics* **50**, 1089 (1982)

Energy and Momentum of particle + EM field system

Conservative field $\rightarrow$ KE + PE (scalar potential) conserved.
EM fields are in general not conservative, so what is conserved?

Expectation: KE of particles + "something" will be conserved.

$$\delta W_M = \int_{all\ vol} \rho(\vec{E}+\vec{v}\times\vec{B}).\vec{v}\,\delta t\, d\tau$$

Work done on the charges=
Force x displacement (integrated over all vol)

$$\frac{dW_M}{dt} = \int \vec{E}.\vec{j}\,d\tau$$

Use $\nabla\times\vec{B} = \mu_0\vec{j}+\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$ to replace $\vec{j}$

$$\nabla.(\vec{E}\times\vec{B}) = \vec{B}.\nabla\times\vec{E} - \vec{E}.\nabla\times\vec{B}$$

$$= \frac{1}{\mu_0}\int(\vec{E}.\nabla\times\vec{B})\,d\tau - \frac{\partial}{\partial t}\int\frac{\epsilon_0 E^2}{2}\,d\tau$$

$$= -\frac{1}{\mu_0}\int\nabla.(\vec{E}\times\vec{B})\,d\tau + \frac{1}{\mu_0}\int\vec{B}.(\nabla\times\vec{E})\,d\tau - \frac{\partial}{\partial t}\int\frac{\epsilon_0 E^2}{2}\,d\tau$$

$$= -\frac{1}{\mu_0}\int\nabla.(\vec{E}\times\vec{B})\,d\tau - \frac{\partial}{\partial t}\int\left(\frac{\epsilon_0 E^2}{2}+\frac{B^2}{2\mu_0}\right)d\tau$$

$$Hence\quad \frac{d}{dt}\left[W_M+\int_{vol}\left(\frac{\epsilon_0 E^2}{2}+\frac{B^2}{2\mu_0}\right)d\tau\right] = -\frac{1}{\mu_0}\int_{surf}\nabla.(\vec{E}\times\vec{B})\,d\tau$$

compare with $\nabla.\vec{j}+\frac{\partial\rho}{\partial t} = 0$

Hence $\frac{dQ_{\text{in}}}{dt} = -\int_{surf}\vec{j}.d\vec{a}$

$$\boxed{\underbrace{\frac{d}{dt}\left[W_M+\int_{vol}\left(\frac{\epsilon_0 E^2}{2}+\frac{B^2}{2\mu_0}\right)d\tau\right]}_{\text{Energy of particles + field}} = -\int_{surf}\underbrace{\frac{1}{\mu_0}(\vec{E}\times\vec{B})}_{\text{Poynting vector}}.d\vec{a}}$$

Energy of particles + field

Poynting vector
The energy flux

Energy and Momentum of particle + EM field system

We find that the EM field contains energy and we can identify the energy flux/flow/current term as well.

Natural question: Can we do the same for momentum of the particles? This is more invloved, becuase momentum is a vector and forming the continuity equation for a vector would require a "tensor".

Apart from that the reasoning is very similar...

Replace charge and current by field terms using Maxwell's equations

$$\frac{d}{dt}\sum_{all}\vec{p}_i = \vec{F} = \int_{all\ vol} \rho(\vec{E}+\vec{v}\times\vec{B})\,d\tau$$

$$= \int\left[\left(\epsilon_0\nabla.\vec{E}\right)\vec{E} + \left(\frac{\nabla\times\vec{B}}{\mu_0}-\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}\right]d\tau$$

$$\text{Since}\ :\ (\nabla\times\vec{B})\times\vec{B} = (\vec{B}.\nabla)\vec{B}-\nabla\frac{B^2}{2}$$

$$\text{And}\ :\ \left(\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\nabla\times\vec{E})$$

$$= \frac{\partial}{\partial t}(\vec{E}\times\vec{B})-\left[(\vec{E}.\nabla)\vec{E}-\nabla\frac{E^2}{2}\right]$$

we have used Faraday's law

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

and a similar expansion again

Energy and Momentum of particle + EM field system

RHS becomes :

$$\epsilon_0\left[(\nabla\,.\,\vec{E})\vec{E}+(\vec{E}\,.\,\nabla)\vec{E}-\nabla\frac{E^2}{2}\right]+\frac{1}{\mu_0}\left[(\nabla\,.\,\vec{B})\vec{B}+(\vec{B}\,.\,\nabla)\vec{B}-\nabla\frac{B^2}{2}\right]-\frac{1}{c^2}\frac{\partial}{\partial t}\frac{(\vec{E}\times\vec{B})}{\mu_0}$$

S=ExB
emerges again

The integrand is now remarkably symmetric in E and B although the initial expression was not. The extra term we have added is div B which is always zero.

$$\frac{d}{dt}\left[\sum_{particles}\vec{p}_i+\frac{1}{c^2}\int\vec{S}\,d\tau\right]=\int\left[\epsilon_0\left\{(\nabla\,.\,\vec{E})\vec{E}+(\vec{E}\,.\,\nabla)\vec{E}-\nabla\frac{E^2}{2}\right\}+\frac{1}{\mu_0}\left\{(\nabla\,.\,\vec{B})\vec{B}+(\vec{B}\,.\,\nabla)\vec{B}-\nabla\frac{B^2}{2}\right\}\right]d\tau$$

Question : Is RHS the divergence of something? Then the form of the continuity equation will emerge again.

But the RHS is already a vector, so it can only be the divergence of tensor (if at all)

Energy and Momentum of particle + EM field system

$$\left[(\nabla.\vec{E})\vec{E}+(\vec{E}.\nabla)\vec{E}-\nabla\frac{E^2}{2}\right]_i$$

$$= \frac{\partial E_j}{\partial x_j}E_i+E_j\frac{\partial E_i}{\partial x_j}-\frac{1}{2}\frac{\partial E^2}{\partial x_i}$$

$$= \frac{\partial}{\partial x_j}\left(E_iE_j-\delta_{ij}\frac{E^2}{2}\right)$$

Repeated index j is summed over, there is no summation over i

Hence the entire RHS integrand is a divergence of the following quantity

$$T_{ij} = \epsilon_0\left(E_iE_j-\delta_{ij}\frac{E^2}{2}\right)+\frac{1}{\mu_0}\left(B_iB_j-\delta_{ij}\frac{B^2}{2}\right)$$

Formally called the Electromagnetic (Maxwell) stress tensor

$$\frac{d}{dt}\left[\sum_{particles}\vec{p}_i+\frac{1}{c^2}\int\vec{S}\,d\tau\right] = -\int_{vol}\nabla.(-\underline{\underline{T}})\,d\tau = -\int_{surf}(-\underline{\underline{T}}).d\vec{a}$$

compare with $\frac{d}{dt}Q_{\text{inside}} = -\int_{vol}\nabla.\vec{j}\,d\tau = -\int_{surf}\vec{j}.d\vec{a}$

Q: Why would you call it a stress tensor?

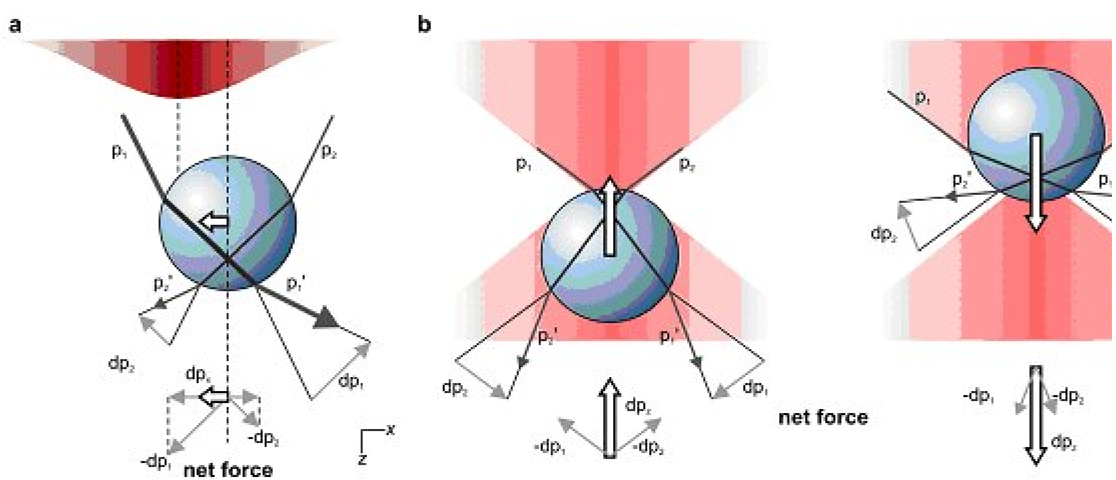
Energy and Momentum of particle + EM field system

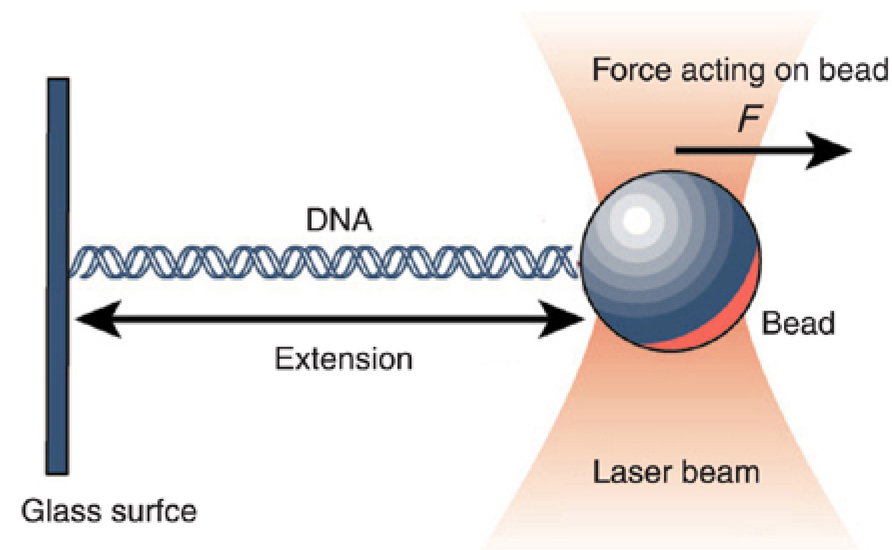
$$\frac{d}{dt}\sum_{particles}\vec{p}_i = \left[-\frac{1}{c^2}\frac{d}{dt}\int\vec{S}\,d\tau + \int_{surf}\underline{T}\,.\,d\vec{a}\right]$$

If we take the volume to include all the particles (or a solid object) then the RHS tells us the total force on that volume.

If there is no t dependence then the integral of T gives the force. In a "mechanical" or "fluid" situation, this is exactly what the stress tensor would have given us.

This formulation can also be used to analyse cases where a focussed beam of light is used to hold up a particle...."optical tweezer"..





Physics world: Optical tweezers: where physics meets biology : Nov 13, 2008

Electromagnetic Waves in free space: What is a plane EM wave ?

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}.\vec{r}-\omega t)}$$
$$\vec{B} = \vec{B}_0 e^{i(\vec{k}.\vec{r}-\omega t)}$$

We start by assuming that Eo and Bo have no spatial dependence, they do NOT depend on x,y,z. All spatial dependence comes from the exponential.

Of course Eo and Bo can have x,y,z components, but they are all constants. These must satisfy Maxwell's equations.

$$\nabla.\vec{E} = 0$$
$$\nabla.\vec{B} = 0$$

$$\nabla . \vec{E}_0 e^{i(k.r-\omega t)}$$
$$= \frac{\partial}{\partial x} E_{0x} e^{i(k.r-\omega t)} + \frac{\partial}{\partial y} E_{0y} e^{i(k.r-\omega t)} \cdots\cdots$$
$$= i(E_{0x}k_x + E_{0y}k_y + E_{0z}k_z)\, e^{i(k.r-\omega t)}$$
$$= i\,\vec{k}.\vec{E}_0\, e^{i(k.r-\omega t)}$$

$$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Similarly Prove that

$$\nabla\times E_0\, e^{i(k.r-\omega t)}$$
$$= i(k\times E_0)\, e^{i(k.r-\omega t)}$$

And $\frac{\partial}{\partial t}\vec{E}_0 e^{i(k.r-\omega t)} = -i\omega \vec{E}_0 e^{i(k.r-\omega t)}$

$$\textit{Hence}\quad \vec{k}.\vec{E}_0 = 0$$
$$\&\quad \vec{k}.\vec{B}_0 = 0$$

We would not have got this "transverse" condition without the assumption of Eo and Bo being constant....in waveguides the condition does NOT hold.

Electromagnetic Waves in free space : What is a plane EM wave?

The third equation gives:

$$\vec{k}\times\vec{E}_0 = \omega\vec{B}_0$$

Hence

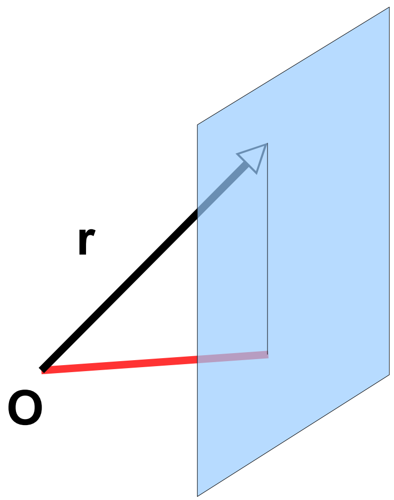
$$\begin{aligned}
\vec{E}_0\times(\vec{k}\times\vec{E}_0) &= \omega\vec{E}_0\times\vec{B}_0 \\
\vec{k}(\vec{E}_0.\vec{E}_0)-\vec{E}_0(\vec{k}.\vec{E}_0) &= \omega\vec{E}_0\times\vec{B}_0 \\
\vec{k} &= \omega\frac{\vec{E}_0\times\vec{B}_0}{E_0^2} \\
|B_0| &= \frac{|E_0|}{c}
\end{aligned}$$

The wave propagates in the direction of E x B.

The relative magnitudes :
For reasonably strong E = 1000V/m
B ~ 3 microTesla very weak .

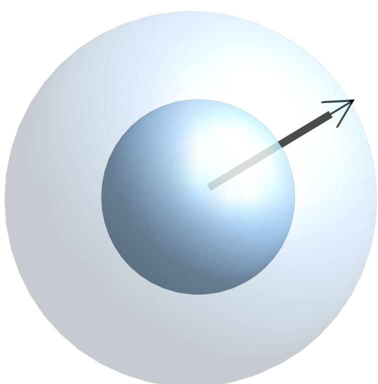
That's why we mostly talk about coupling with the electric field of light.

Electromagnetic Waves in free space : Wavefronts and their shapes



Wavefront of plane waves

Plane normal to **k**
k.r = length of the red line x magnitude of k as long as the tip of **r** lies in the plane.
Surfaces of constant **k.r** at a certain time t are called wavefronts.
For plane waves the wavefronts are planes.
For sphereical waves these would be spherical surfaces.



Simple spherical wavefront described by

$$V(r,t) = \frac{A}{r} e^{i(kr-\omega t)}$$

It is NOT $\vec{k}.\vec{r}-\omega t$

Any wave coming from a source (like light from a point) is in reality spherical. But at large distances it is approximated by a plane wave very well.

This is similar to neglecting the earth's curvature over a small region....

Electromagnetic Waves in free space :Energy, momentum density and Intensity

$$\vec{E}(r,t) = \vec{E}_0 \cos(\vec{k}.\vec{r} - \omega t)$$

$$Hence \ \langle E^2 \rangle = \frac{1}{T}\int_0^T E_0^{\,2} \cos^2(\vec{k}.\vec{r} - \omega t)\, dt$$

$$= \frac{E_0^{\,2}}{2}$$

$$Energy \quad U = \left(\frac{\epsilon_0 \langle E^2 \rangle}{2} + \frac{\langle B^2 \rangle}{2\mu_0} \right) = \frac{\epsilon_0 E_0^{\,2}}{2}$$

Using the earlier results

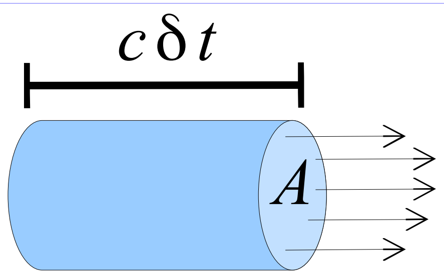
$$|B| = \frac{|E|}{c}$$

$$\epsilon_0 \mu_0 = \frac{1}{c^2}$$

Although the B field is much weaker, E and B components make equal contributions to the field energy.

$$Momentum \ \vec{p} = \frac{\vec{S}}{c^2} = \frac{1}{\mu_0 c^2} \langle \vec{E} \times \vec{B} \rangle$$

$$|p| = \frac{U}{c}$$



$$Intentsity \quad I = \frac{A(c\delta t)U}{A\delta t} = Uc$$

Intensity : Energy passing through per unit area per unit time.

All the energy in the volume will pass through the cross section in time dt

Maxwell's equation in "linear" matter : what happens to the wave equation?

We consider an insulator first, so there are no free charges in the material

$$\vec{D} = \epsilon \vec{E} \qquad \nabla . \vec{D} = 0$$
$$\vec{B} = \mu \vec{H} \qquad \nabla . \vec{B} = 0$$

But now both magnetisation and electric polarisation can simultaneously change. So the "bound" current will result from change in **M** as well as **P**.

$\sigma_b = \vec{P} . \hat{n}$: Then consider $\vec{P} \rightarrow \vec{P} + \delta \vec{P}$

This change causes some amount of charge to flow in/out

$$\delta Q = \delta(\vec{P}.\hat{n}) \delta a$$
$$\vec{J}_p . \delta \vec{a} = \frac{\delta Q}{\delta t} = \frac{\partial \vec{P}}{\partial t} . \delta \vec{a}$$

Total bound current flow

$$\boxed{\vec{J}_b = \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t}}$$

Show that this interpretation is consistent with the continuity equation

$$\nabla \times \vec{B} = \mu_0 \vec{J}_{total} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$
$$\nabla \times [\mu_0 (\vec{H} + \vec{M})] = \mu_0 \left[\vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right] + \mu_0 \frac{\partial}{\partial t} [\vec{D} - \vec{P}]$$
$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Maxwell's equation in "linear" matter : what happens to the wave equation?

$$\nabla \cdot \vec{D} = 0$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{B} = \mu_0 (\vec{H} + \vec{M})$$
$$\epsilon_0 \vec{E} = (\vec{D} - \vec{P})$$

$$\vec{D} = \epsilon \vec{E}$$
$$\vec{B} = \mu \vec{H}$$

With $\vec{J}_f = 0$ *we will get* $\nabla \times \vec{B} = \mu \epsilon \frac{\partial \vec{E}}{\partial t}$

The wave will propagate with speed $v^2 = \frac{1}{\mu \epsilon}$

Refractive index of the medium $n = \frac{c}{v} = \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}}$

Maxwell's equation in "linear" matter : The boundary conditions

Consider a boundary between two media 1 and 2

Since div D = 0, the normal component of D must be continuous.
div B = 0, always (so normal component of B is continuous)

Since curl H has no singularities … the tangential component of H is continuous
curl E has no singularities … the tangnetial component of E is continuous

$$
\begin{array}{llll}
D_1^{\perp} = D_2^{\perp} & \quad Hence & \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \\
B_1^{\perp} = B_2^{\perp} & & \\
H_1^{\parallel} = H_2^{\parallel} & \quad Hence & \dfrac{B_1^{\parallel}}{\mu_1} = \dfrac{B_2^{\parallel}}{\mu_2} \\
E_1^{\parallel} = E_2^{\parallel} & &
\end{array}
$$

These boundary conditions govern the reflection and transmission of electromagnetic waves at an interface and hence the laws of reflection and refraction (optics)

Electromagnetic waves at an interface : reflection and transmission

Normal incidence

x=0

1 | 2

The incident wave propagating to the right

$$\vec{E}_I = E_{0\mathrm{I}}\, e^{i(k_1 x - \omega t)}\,\hat{y}$$

$$\vec{B}_I = \frac{1}{v_1} E_{0\mathrm{I}}\, e^{i(k_1 x - \omega t)}\,\hat{z}$$

The reflected wave propagating to the left

$$\vec{E}_R = E_{0\mathrm{R}}\, e^{i(-k_1 x - \omega t)}\,\hat{y}$$

$$\vec{B}_R = -\frac{1}{v_1} E_{0\mathrm{R}}\, e^{i(-k_1 x - \omega t)}\,\hat{z}$$

The transmitted wave propagating to the right

$$\vec{E}_R = E_{0\mathrm{R}}\, e^{i(k_2 x - \omega t)}\,\hat{y}$$

$$\vec{B}_R = \frac{1}{v_2} E_{0\mathrm{R}}\, e^{i(k_2 x - \omega t)}\,\hat{z}$$

Tangential E
Tangential H
are continuous

$$E_{0\mathrm{I}} + E_{0\mathrm{R}} = E_{0\mathrm{T}}$$

$$\frac{1}{\mu_1}\left(\frac{E_{0\mathrm{I}}}{v_1} - \frac{E_{0\mathrm{R}}}{v_1}\right) = \frac{1}{\mu_2}\frac{E_{0\mathrm{T}}}{v_2}$$

$$define \qquad \beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

Need to solve for the ratios only....

$$\frac{E_{0\mathrm{R}}}{E_{0\mathrm{I}}} = \frac{1-\beta}{1+\beta} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right|$$

$$\frac{E_{0\mathrm{T}}}{E_{0\mathrm{I}}} = \frac{2}{1+\beta} = \left(\frac{2 n_1}{n_1 + n_2}\right)$$

An useful result with three "phasor" s:

$$A\,e^{iax} + B\,e^{ibx} = C\,e^{icx} \quad \forall \quad x$$

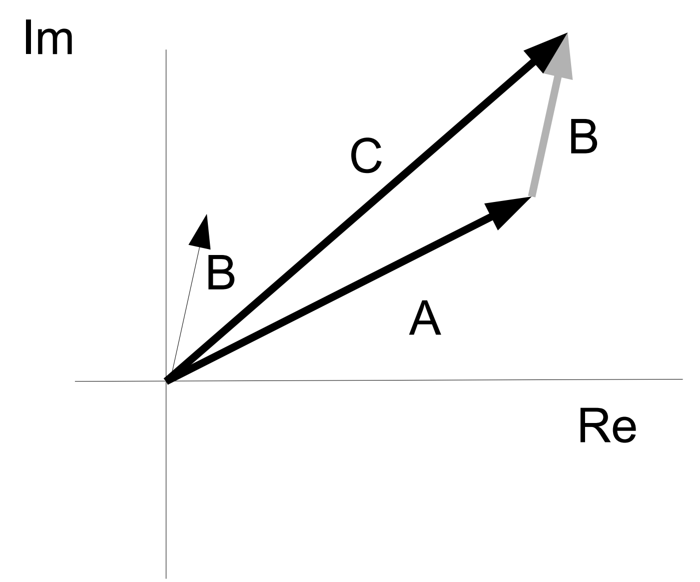
$$Then \quad a = b = c$$

$$set\, x = 0 \quad : \quad this\ gives \quad A + B = C$$

This condition determines the length of the phasors, which must be satisfied at all times

$$Now \quad draw \quad the \quad three \quad phasors \quad when \quad x \neq 0$$

Two sides of a traingle are together greater than the third side



The equality can only hold if A, B, C *are along the same ray..*

The phase angle also must be same

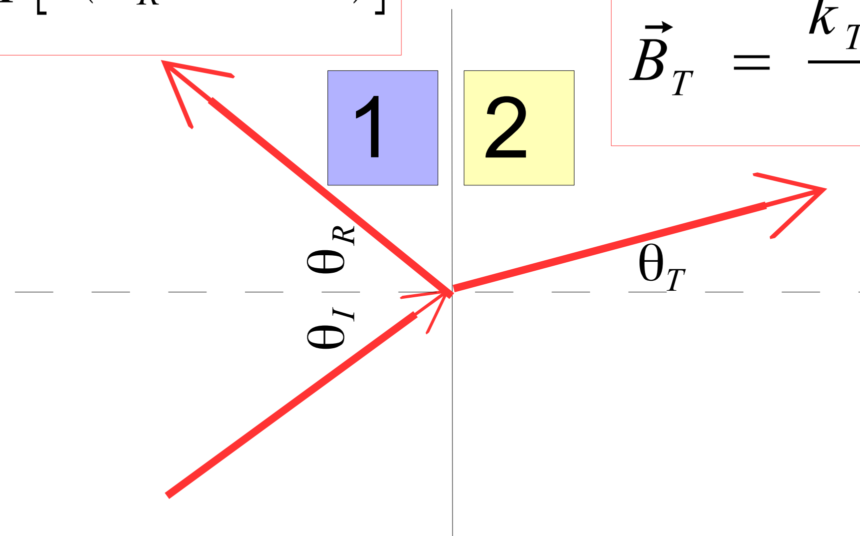
$$implies \quad a = b = c$$

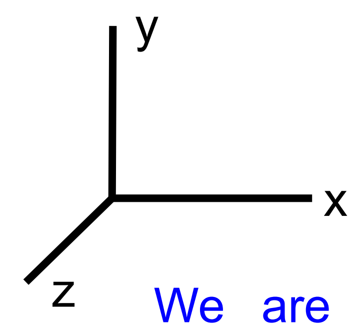
Electromagnetic waves at an interface : reflection and refraction

$$\vec{E}_R = \vec{E}_{0R}\exp\left[i\left(\vec{k}_R.\vec{r}-\omega t\right)\right]$$
$$\vec{B}_R = \frac{\hat{k}_R\times\vec{E}_{0R}}{v_1}\exp\left[i\left(\vec{k}_R.\vec{r}-\omega t\right)\right]$$

Oblique incidence at an interface (general case)

$$\vec{E}_T = \vec{E}_{0T}\exp\left[i\left(\vec{k}_T.\vec{r}-\omega t\right)\right]$$
$$\vec{B}_T = \frac{\hat{k}_T\times\vec{E}_{0T}}{v_2}\exp\left[i\left(\vec{k}_T.\vec{r}-\omega t\right)\right]$$





We are looking at the x=0 plane sideways

$$\vec{E}_I = \vec{E}_{0I}\exp\left[i\left(\vec{k}_I.\vec{r}-\omega t\right)\right]$$
$$\vec{B}_I = \frac{\hat{k}_I\times\vec{E}_{0I}}{v_1}\exp\left[i\left(\vec{k}_I.\vec{r}-\omega t\right)\right]$$

Notice how unit vectors have been used to fix the relative directions

$$\omega=\left|\vec{k}\right|v \quad : \quad Hence \quad k_I v_1 = k_R v_1 = k_T v_2$$

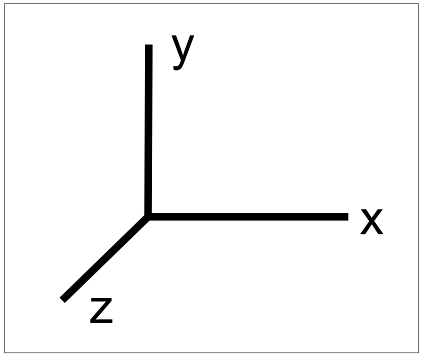
Use the result derived just before

$$\vec{k}_I.\vec{r} = \vec{k}_R.\vec{r} = \vec{k}_T.\vec{r} \quad must \quad hold \quad \forall\, r \quad \text{on}\, the\; x=0\; plane$$

Electromagnetic waves at an interface : reflection and refraction

$$k_I = k_R = \frac{v_2}{v_1} k_T \qquad \text{in magnitude}$$

$$\left.\begin{aligned}(k_I)_y y + (k_I)_z z &= (k_R)_y y + (k_R)_z z \\ (k_I)_y y + (k_I)_z z &= (k_T)_y y + (k_T)_z z\end{aligned}\right\} \quad holds \;\; \forall \;\; y, z$$



This means all the coefficients (y,z components) must be equal

Form the triple product of k_I, k_R, k_T : this must vanish since two row/columns are identical.

The three vectors are co-planer [Law of reflection and refraction]
Let this be the x-y plane.

Since $|\mathbf{k}_I| = |\mathbf{k}_R|$ and y components are equal, the other (x) component is exactly reversed.
No other possibility can satisfy all these conditions.

Equality of the y-components require

$$\theta_I = \theta_R$$

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$